Solving the exact background Collisional Boltzmann Equation (CBE) for DM-baryon scattering

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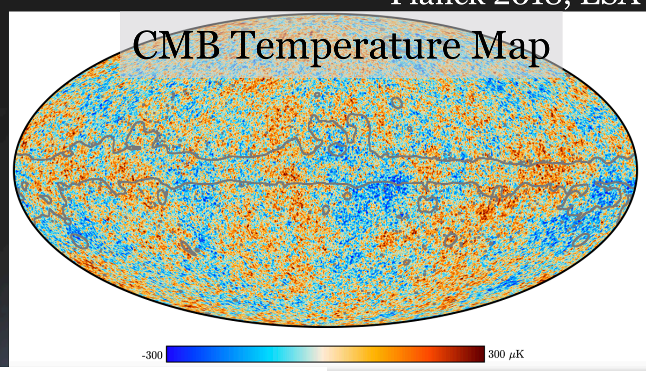
Parallel Talk
Cosmology from Home (2022)

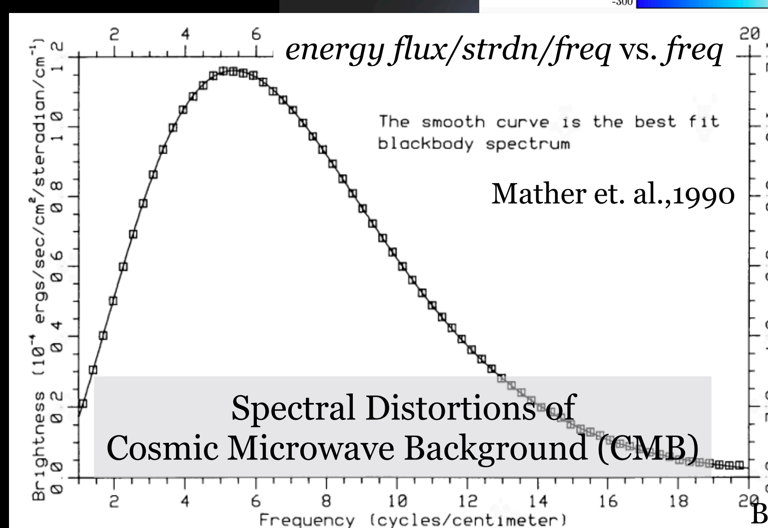
Paper on arXiv: 2205.05536 (submitted to Phys. Rev. D)

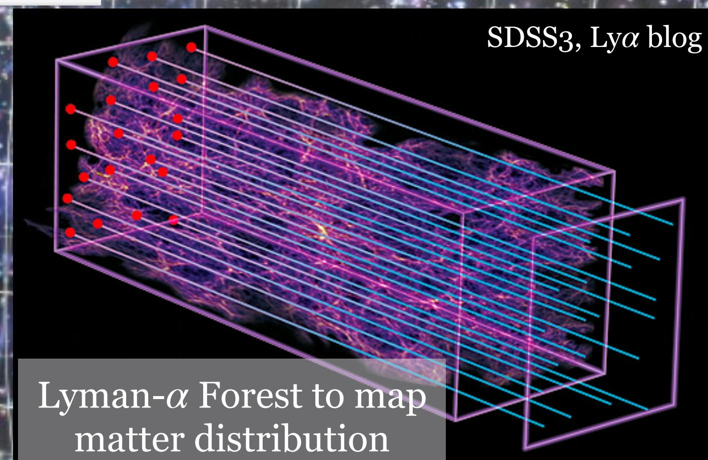
A Few Cosmological Probes of DM-baryon (χ-s) Scattering

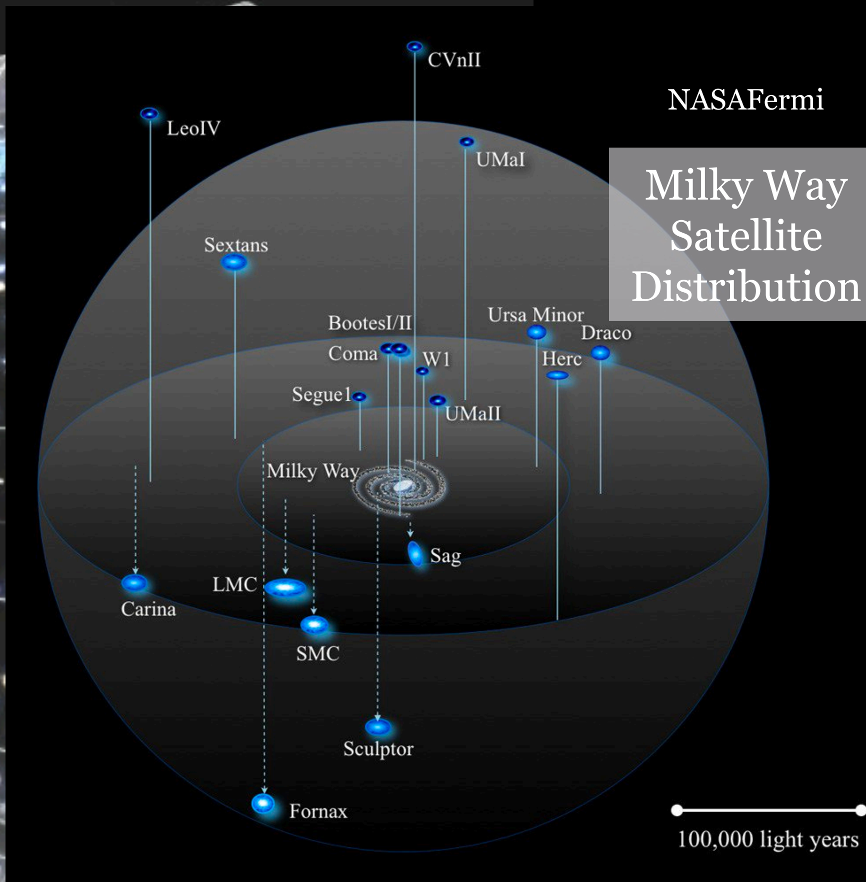
Planck 2018, ESA



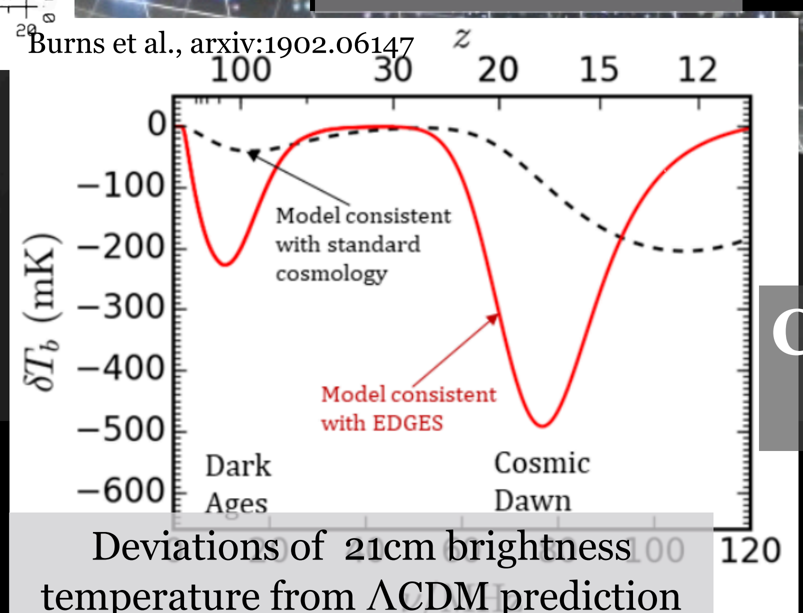
time







Burns et al., arxiv:1902.06147



Constraints from all of these probes rely on a crucial assumption…

$$f_\chi(\overrightarrow{v}) \underset{(?)}{=} f_\chi^{\mathrm{MB}}(\overrightarrow{v})$$

- ***Almost all*** current constraints on χ-s scattering cross-section $\sigma_{\chi s}$ ***assume*** $f_\chi(\overrightarrow{v}) = f_\chi^{\mathrm{MB}}(\overrightarrow{v})$

  $\Rightarrow$ $\dot{Q}_\chi$, $\dot{\overrightarrow{p}}_\chi$: ***analytical*** heat & momentum transfer rates

  $\Rightarrow$ No need to solve the Boltzmann eq.:

$$\mathrm{d}f_\chi/\mathrm{d}t = C_{\chi s}[f_\chi] + C_{\chi\chi}[f_\chi]$$

- But, if χ-χ scattering is ineff., $f_\chi(\overrightarrow{v}) \neq f_\chi^{MB}(\overrightarrow{v})$ after χ-s decoupling

- If $f_\chi = f_\chi^{\mathrm{MB}} \Rightarrow C_{\chi\chi}[f_\chi^{\mathrm{MB}}] = 0$, eliminates the analysis of DM self-interactions

$f_\chi(\overrightarrow{v}) \underset{(?)}{\neq} f_\chi^{\mathrm{MB}}(\overrightarrow{v})$

- If $f_\chi(\overrightarrow{v}) \neq f_\chi^{MB}(\overrightarrow{v})$, then $\dot{Q}_\chi$, $\dot{\overrightarrow{p}}_\chi$ are no longer analytical
- Need to implement the *full* collision operator

out of $\overrightarrow{v}'$, into $\overrightarrow{v}$ — out of $\overrightarrow{v}$, into $\overrightarrow{v}'$

$$C_{\chi s}[f_\chi](\overrightarrow{v}) = \int d^3v' \left(\boxed{\Gamma_{\chi s}(\overrightarrow{v}' \to \overrightarrow{v}) f_\chi(\overrightarrow{v}')} - \boxed{\Gamma_{\chi s}(\overrightarrow{v} \to \overrightarrow{v}') f_\chi(\overrightarrow{v})} \right)$$

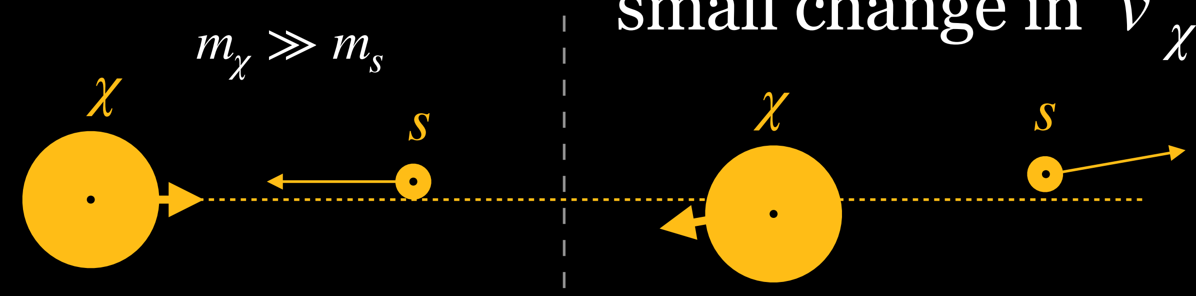
- $\Gamma_{\chi s}(\overrightarrow{v} \to \overrightarrow{v}')$: χ-s scattering rate from DM velocity $\overrightarrow{v}$ to $\overrightarrow{v}'$, is generally a 5D non-analytic integral
- Highly non-trivial calculations, incompatible with standard CMB codes

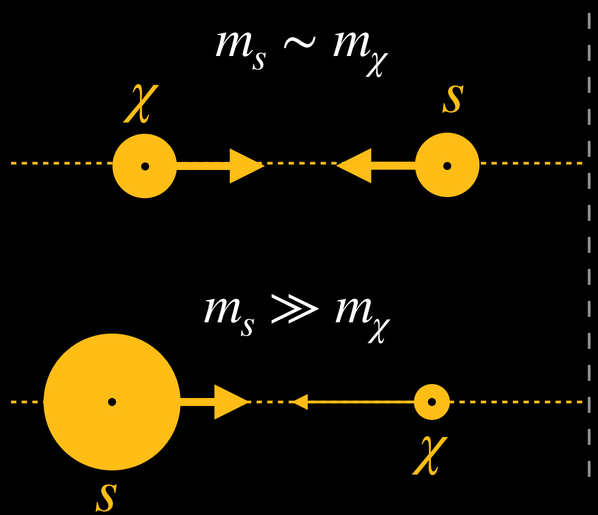
Motivation: can analyze DM self-interactions, since

$$C_{\chi\chi}[f_\chi] \neq 0$$

Previous Work: Fokker-Planck (FP) Approximation

Diffusive scattering: small change in $\vec{v}_\chi$





Is scattering diffusive?

Which is more accurate, FP approx. or MB assumption?

- The FP (or *diffusion*) approximation to the full CBE was formulated in Ali-Haïmoud (2019)
- Numerically more tractable
- Scattering is diffusive when $\Delta \vec{v}_\chi$ is small
- Need an exact method to determine the accuracy of the diffusion formalism for $m_\chi \lesssim m_s$

Finding an *exact* solution to the *background* CBE

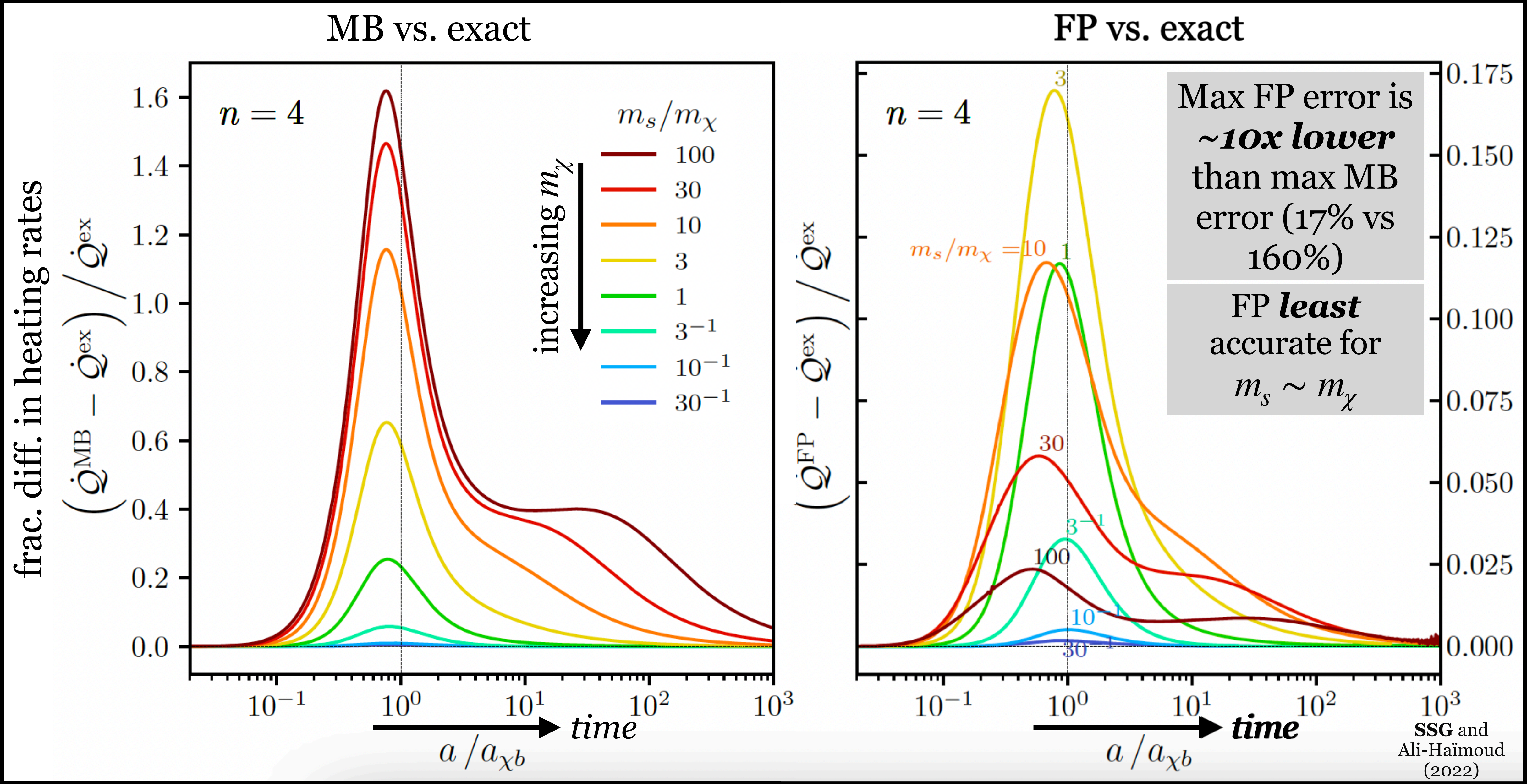
- Homogeneity & Isotropy:
  $\mathscr{F}_\chi(\vec{x}, \vec{v}) = \bar{n}_\chi \bar{f}_\chi(v) = \bar{n}_\chi f_\chi^{1D}(v) \big/ \left(4\pi v^2\right)$
- Do not account for χ-χ scattering (quadratic in $\bar{f}_\chi$), and calculate the ***maximal*** error due to the MB and FP methods
- Need to implement the 1D χ-s collision operator:

$$C_{\chi s}^{1D}[f_\chi](v) = \int dv' \left(\Gamma_{\chi s}^{1D}(v' \to v) f_\chi^{1D}(v') \; - \; \Gamma_{\chi s}^{1D}(v \to v') f_\chi^{1D}(v) \right)$$

- We show that $\Gamma_{\chi s}^{1D}(v \to v')$—a 4D integral—is:
  1. Reducible to a ***1D integral*** for ***isotropic*** differential cross-sections $d\sigma_{\chi s} \big/ d\Omega$,
  2. ***Fully analytical*** if $d\sigma_{\chi s} \big/ d\Omega \propto v_{\chi s}^n, \quad n \in \{0, 2, 4, \dots\}$
- Solve $df_\chi^{1D} \big/ dt = C_{\chi s}^{1D}[f_\chi^{1D}]$ and obtain $f_\chi^{1D}(v, t)$!

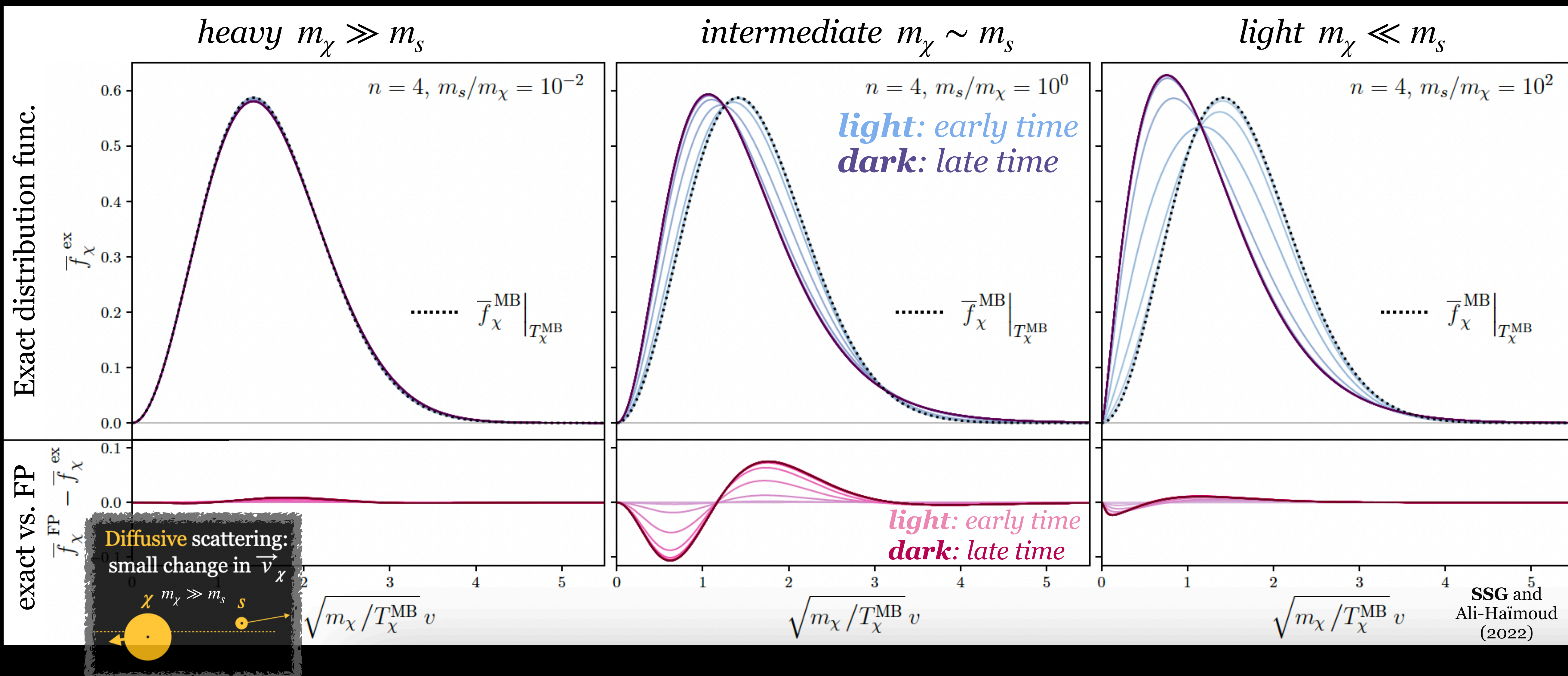
Results: Comparing $\dot{Q}_\chi^{\rm MB}$ and $\dot{Q}_\chi^{\rm FP}$ to $\dot{Q}_\chi^{\rm ex}$

- Use heating rate $\dot{Q}_\chi$ to compare MB, FP, and exact methods
- Showing fractional differences in $\dot{Q}_\chi$ for a range of mass ratios:



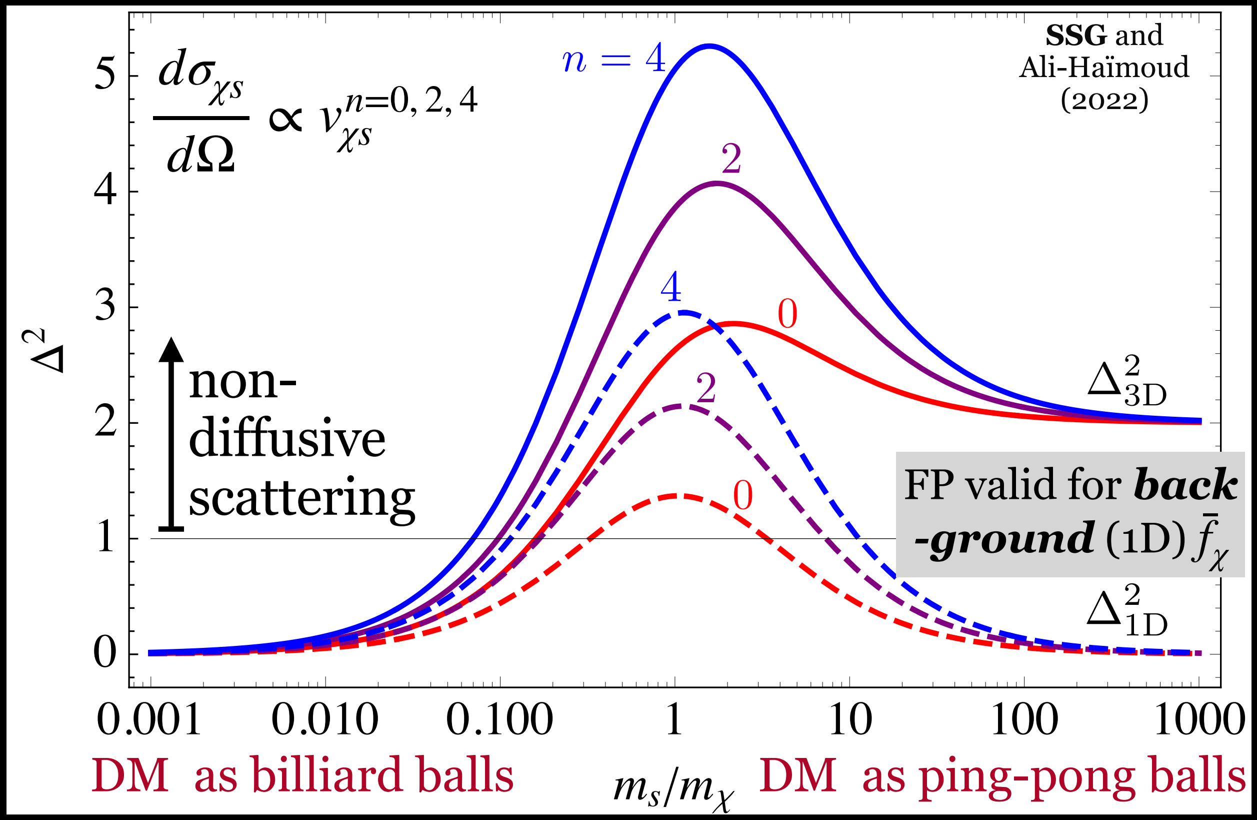
Results: Comparing $\bar{f}_\chi^{\rm MB}$ and $\bar{f}_\chi^{\rm FP}$ to $\bar{f}_\chi^{\rm ex}$

We can understand the $\dot{Q}_\chi$ results by comparing the different $\bar{f}_\chi$:



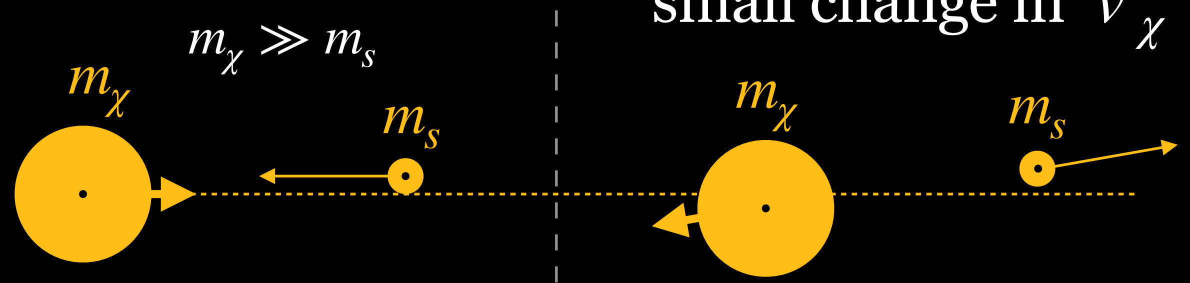
Is FP accurate for $m_\chi \ll m_s$?
– 3D vs. 1D diffusion

- 3D diffusion requires small change in DM vel. *vector* $\vec{v}$
- 1D diffusion only requires small change in DM vel. *magnitude* v
- Showing coefficients that quantify diffusivity of 3D & 1D scattering for given m_s/m_χ
- Scattering is non-diffusive for $m_s \sim m_\chi$ in ***both*** 1D and 3D
- FP accuracy $\lesssim 17\,\%$ in 1D for $m_s \sim m_\chi$ bodes well for 3D

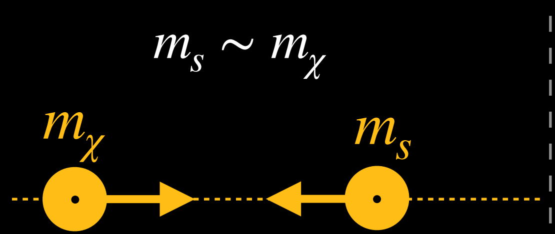


3D diffusion is relevant for CMB analyses in the presence of cosmological perturbations

Takeaways



Diffusive scattering: small change in $\vec{v}_\chi$



Is scattering diffusive? ***— No***

Which is more accurate, FP approx. or MB assumption?

— FP has higher accuracy

- The FP approximation is valid for *heavier* DM,
- But even when it is physically invalid, it recovers the correct $\dot{Q}_\chi$ within ~17%—***10x better than MB*** in 1D
- Indicative of ***better accuracy with FP***, even in 3D—relevant for CMB constraints on $\sigma_{\chi s}$
- ***Integrating FP*** formalism (as opposed to exact) ***with CMB codes*** is a more tractable task
- Importantly, FP approx. would ***allow for DM self-scattering*** (not possible with MB, and intractable with exact)

Thank you for watching!

*My **research interests** include:*

- Effects of DM-baryon interactions on cosmological observables

- (Relatively) Late-time ($z \lesssim 10^3$) χ-s interaction models

- Reassessing the standard mathematical framework for CMB constraints on $\sigma_{\chi s}$ (besides the MB assumption)

- Stellar and galactic dynamics

***Comments**, **questions**, and any **feedback** is very welcome!*

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Fokker-Planck Method

- Computes the correct heating and momentum-exchange rates for a *given* f_χ
- However, the f_χ that the FP collision operator determines is only an approximation for the true f_χ

  — so the resultant rates might not be accurate
- Turns CBE into PDE
- Discretization leads to sparse, tri-diagonal collision matrix, as opposed to full.

Mass-dependence of FP validity

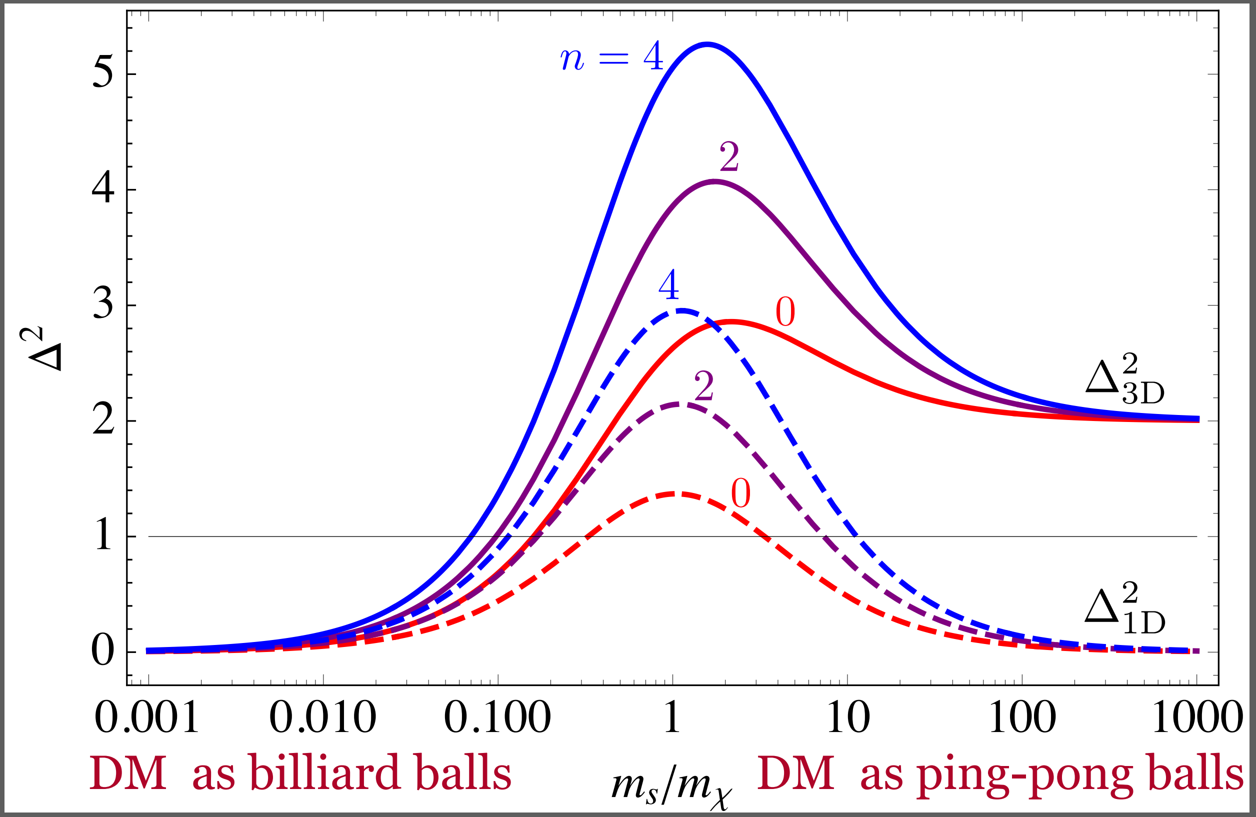
$$\Delta^2_{3D}(v^2) \equiv \frac{\left\langle \sigma_{\chi s}(v_{\chi s})\, v_{\chi s} \left| \vec{v}\,' - \vec{v} \right|^2 \right\rangle}{\left\langle \sigma_{\chi s}(v_{\chi s}) v_{\chi s} \right\rangle v^2}$$

$$\langle (\vec{v}\,' - \vec{v})^2 \rangle_{\hat{n}'} = 2\left(\frac{m_s}{M}\right)^2 |\vec{v} - \vec{v}_s|^2$$

Assuming fwd-bckwd scattering symmetry, we can avg out the $\vec{v}\,'$ dependence

$$\Delta^2_{1D}(v^2) \equiv \frac{\left\langle \sigma_{\chi s}(v_{\chi s})\, v_{\chi s} \left(|\vec{v}\,'|^2 - |\vec{v}|^2 \right) \right\rangle}{\left\langle \sigma_{\chi s}(v_{\chi s}) v_{\chi s} \right\rangle v^2}$$

$$\langle v'^2 - v^2 \rangle_{\hat{n}'} = 2\left(\frac{m_s}{M}\right)^2 |\vec{v} - \vec{v}_s|^2 - 2\frac{m_s}{M}(\vec{v} - \vec{v}_s)\cdot\vec{v}$$



By triangle inequality,

$$\Delta|\vec{v}| \leq |\Delta\vec{v}| \Rightarrow \Delta^2_{1D} \leq \Delta^2_{3D}$$

Simplifying $\Gamma_{\chi s}$

[which of these is the first simplification that allows us to factorize $\Gamma_{\chi s}$ into (time dep.) $\times\ \widetilde{\Gamma}$?] –the fact that diff cross-sec is a power-law in $v_{\chi s}$! Doesn't have to be isotropic– it can also be a func of $\hat{n}\cdot\hat{n}'$. We only need to be able to pull out the factors of T_b/m_s (or T_b/m_χ)

- Start with the full 3D expression:

$$\Gamma(\boldsymbol{v}\to\boldsymbol{v}') = \int d^3v_s\ f(\boldsymbol{v}_s)\ v_{\chi s} \int d^2\hat{n}'\ \frac{d\sigma}{d^2\hat{n}'}\ \delta^{(3)}\left[\boldsymbol{v}' - \boldsymbol{v} - R(v_{\chi s}\hat{n}' - \boldsymbol{v}_{\chi s})\right], \qquad \boldsymbol{v}_{\chi s} \equiv \boldsymbol{v} - \boldsymbol{v}_s, \qquad R \equiv \frac{m_s}{M}$$

- Assume (any) isotropic differential cross-section, and split $\delta_D^{(3)}$ into $\delta_D^{(2)}$ and $\delta_{\mathrm{D}}^{(1)}$. The integral simplifies to 3D, with a 1D δ_D function:

$$\Gamma(\boldsymbol{v}\to\boldsymbol{v}') = \frac{\sigma_n}{2\pi R}\int d^3v_{\chi s} f(\boldsymbol{v} - \boldsymbol{v}_{\chi s}) v_{\chi s}^n\ \delta^{(1)}[v^2 + v'^2 - 2R\boldsymbol{v}\cdot\boldsymbol{v}_{\chi s} + 2(R\boldsymbol{v}_{\chi s} - \boldsymbol{v})\cdot\boldsymbol{v}'].$$

- Now if $f_s(\overrightarrow{v}_s)$ is isotropic in one unique frame, $\Gamma(\overrightarrow{v} \to \overrightarrow{v}')$ only on $\left|\overrightarrow{v} - \overrightarrow{V}_b\right|$, $\left|\overrightarrow{v}' - \overrightarrow{V}_b\right|$, and $\hat{v}\cdot\hat{v}'$.

  This is useful if we want to find $\int d^2v' \int \frac{d^2v}{4\pi}\Gamma(\overrightarrow{v} \to \overrightarrow{v}')$, as it renders one of these integrals redundant.

- $\int d^2v' \int \frac{d^2v}{4\pi}\Gamma(\overrightarrow{v} \to \overrightarrow{v}')$ can then be reduced to a 1D integral over $v_{\chi s}$.
- For $\frac{d\sigma_{\chi s}}{d^2\hat{n}'} = \frac{\sigma_n v_{\chi s}^n}{4\pi}$, it is reducible to a completely analytical expression for $n \in \{0, 2, 4, \dots\}$

Solving the 1D CBE exactly

- For the first time, we find fully analytical expressions for $\Gamma^{1D}_{\chi s}(v \to v')$ with $\dfrac{d\sigma_{\chi s}}{d^2\hat{n}'} \propto v^n_{\chi s}, \quad n \in \{0, 2, 4, \ldots\}$
- Recasting velocities as $u \equiv \sqrt{m_\chi / T_b}\, v$, we can factorize the problem:

$$\underset{\text{(Collision Operator)}}{\mathbf{C}_i} = \underset{\text{(invariant scattering matrix)}}{\tilde{\mathbf{M}}_{ij}} \times f(u_i) \times \text{time dep. factor}$$

- Convert *integro-differential CBE* into a set of *coupled ODEs*—much simpler to solve numerically

Steps from integro-diffrn. eq to coupled ODEs

Integro-differential equation:

$$a^{1/2}\, \frac{d}{dt}\bigg|_{\rm free} [a^{-1/2}\widetilde{f}(u)] = R_n(T_b)\; \widetilde{C}[\widetilde{f}](u), \;\rightarrow\; \frac{\partial \Delta\widetilde{f}}{\partial \ln a} = \frac{R_n(T_b)}{H}\widetilde{C}[\Delta\widetilde{f}] + \frac{1}{2}\frac{\partial}{\partial u}\left[u(\widetilde{f}^{\rm eq} + \Delta\widetilde{f})\right].$$

Define rescaled time var., $y = a/a_{\chi b}$ and get:

$$\frac{\partial \Delta\widetilde{f}}{\partial \ln y} = y^{-\frac{n+3}{2}}\widetilde{C}[\Delta\widetilde{f}] + \frac{1}{2}\frac{\partial}{\partial u}\left[u(\widetilde{f}^{\rm eq} + \Delta\widetilde{f})\right].$$

$$a_{\chi b}^{\frac{n+3}{2}} \equiv 2c_n\sigma_n \frac{n_{s,0}}{H_0\Omega_r^{1/2}}\left(\frac{M^2}{m_s m_\chi}\right)^{\frac{n-1}{2}}\left(\frac{T_{r,0}}{M}\right)^{\frac{n+1}{2}}$$

$$\tilde{f}(u) \equiv \sqrt{T_b/m_\chi}\; f^{\rm 1D}(v)$$

$$\Delta\tilde{f}(u) \equiv \tilde{f}(u) - \tilde{f}^{\rm eq}(u)$$

1D collision operator:

$$\widetilde{C}[\Delta\widetilde{f}](u_i) = \sum_{j=0}^{N-1} M_{ij}\Delta\widetilde{f}_j,$$

$$M_{ij} \equiv d\ln u\left(\widetilde{\Gamma}_{ji}\,u_j - \delta_{ij}\sum_k \widetilde{\Gamma}_{ik}u_k\right)$$

Numerical gradient operator:

$$\frac{\partial(u\widetilde{f})}{\partial u}\bigg|_i = \frac{1}{2u_i d\ln u} \times \begin{cases} (u_0\widetilde{f}_0 + u_1\widetilde{f}_1) & i=0 \\ \left(u_{i+1}\widetilde{f}_{i+1} - u_{i-1}\widetilde{f}_{i-1}\right) & 1\le i\le N-2 \\ -(u_{N-2}\widetilde{f}_{N-2} + u_{N-1}\widetilde{f}_{N-1}) & i=N-1 \end{cases}$$
$$\equiv \sum_{j=i-1}^{i+1}\alpha_{ij}\widetilde{f}_j, \quad \text{where} \quad \alpha_{0,-1} = \alpha_{N-1,N} = 0.$$

Final eq. with coupled ODEs:

$$\frac{\partial \Delta\widetilde{f}_i}{\partial \ln y} = y^{-\frac{n+3}{2}}\sum_{j=0}^{N-1} M_{ij}\Delta\widetilde{f}_j + \frac{1}{2}\sum_{j=i-1}^{i+1}\alpha_{ij}(\widetilde{f}_j^{\rm eq} + \Delta\widetilde{f}_j)$$

Equations for a_{dec}, $\dot{Q}_\chi^{MB}$

If $f = f^{MB}\Big|_{T_\chi^{MB}}$:

$$\dot{\mathcal{Q}}_\chi = \frac{\rho_\chi \rho_s}{M^2 v_{\rm th}^2}\mathcal{B}(V_{\chi b}; v_{\rm th}^2) \times (T_b - T_\chi) + \frac{\rho_\chi \rho_s}{M v_{\rm th}^2}\frac{T_\chi}{m_\chi}\mathcal{A}(V_{\chi b}; v_{\rm th}^2) V_{\chi b}^2.$$

$$\mathcal{A}(w; T/m) = c_n\ \sigma_n \left(\frac{T}{m}\right)^{\frac{n+1}{2}} \alpha_n(\sqrt{m/T}\ w), \quad (36)$$

$$\mathcal{B}(w; T/m) = 3c_n\ \sigma_n \left(\frac{T}{m}\right)^{\frac{n+3}{2}} \beta_n(\sqrt{m/T}\ w), \quad (37)$$

$$\alpha_n(x) \equiv\ {}_1F_1\left(-\frac{n+1}{2}, \frac{5}{2}, -\frac{x^2}{2}\right), \quad (38)$$

$$\beta_n(x) \equiv\ {}_1F_1\left(-\frac{n+3}{2}, \frac{3}{2}, -\frac{x^2}{2}\right), \quad (39)$$

$$c_n \equiv \frac{2^{\frac{5+n}{2}}}{3\sqrt{\pi}}\Gamma(3+n/2). \quad (40)$$

Ali-Haïmoud 2019

$$\dot{T}_\chi = -2HT_\chi + \Gamma_{\chi b}(T_b - T_\chi), \quad (1)$$

$$\text{with} \quad \Gamma_{\chi b} \equiv \frac{2c_n N_b \sigma_n m_b m_\chi}{(m_b + m_\chi)^2}\left(\frac{T_b}{m_b} + \frac{T_\chi}{m_\chi}\right)^{\frac{n+1}{2}}, \quad (2)$$

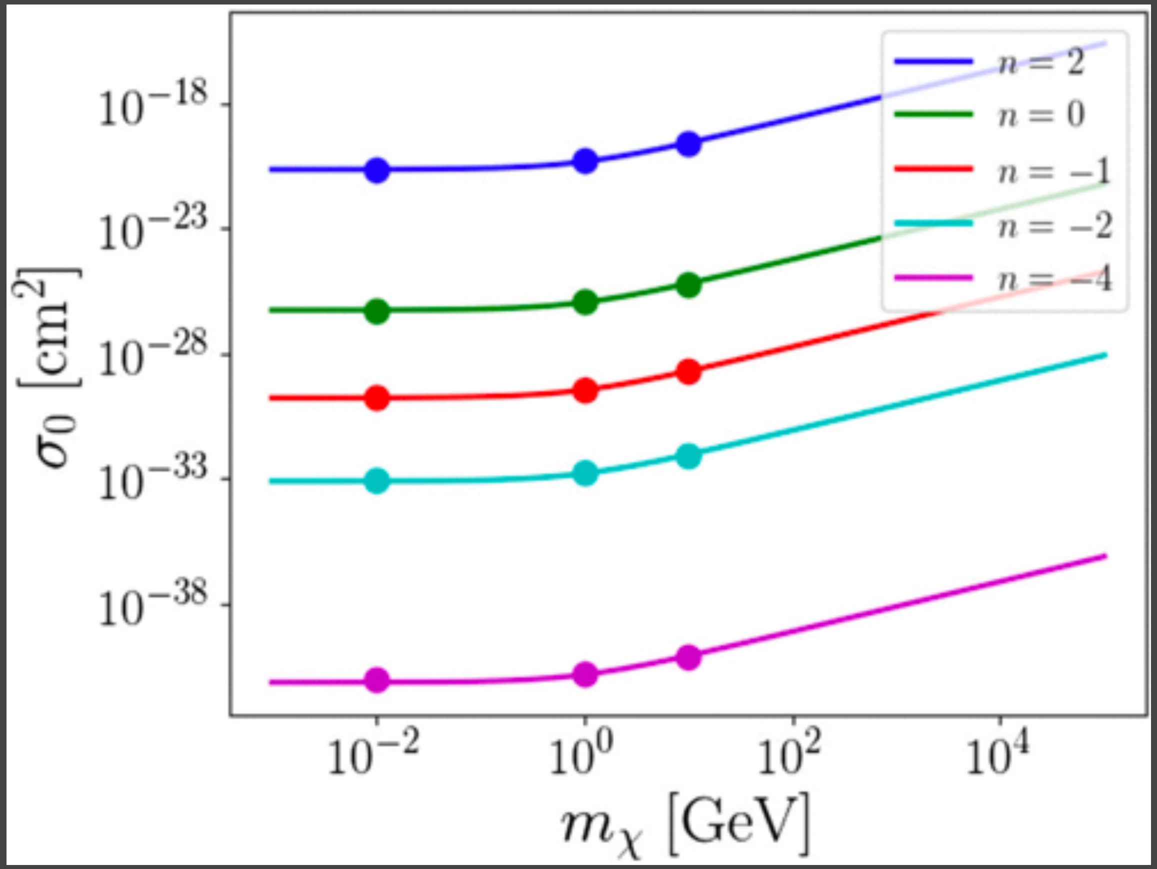
$$\Gamma_{\chi b}^{eq(2)} \sim \frac{m_\chi}{m_\chi + m_b}\Gamma_{\chi b,tot} \propto v^{n+1}$$

$$\frac{\Gamma_{\chi b}}{H} = \left(\frac{a_{\chi b}}{a}\right)^{\frac{n+3}{2}}\left(\frac{m_\chi/m_b + T_\chi/T_b}{m_\chi/m_b + 1}\right)^{\frac{n+1}{2}}, \quad (3)$$

$$(a_{\chi b})^{\frac{n+3}{2}} \equiv \frac{m_b}{m_\chi}\left(1 + \frac{m_b}{m_\chi}\right)^{\frac{n-3}{2}} \frac{2c_n\sigma_n N_b^0\left(\frac{T_\gamma^0}{m_b}\right)^{\frac{n+1}{2}}}{H_0(\Omega_r^0)^{1/2}}. \quad (4)$$

Ali-Haïmoud et al 2015

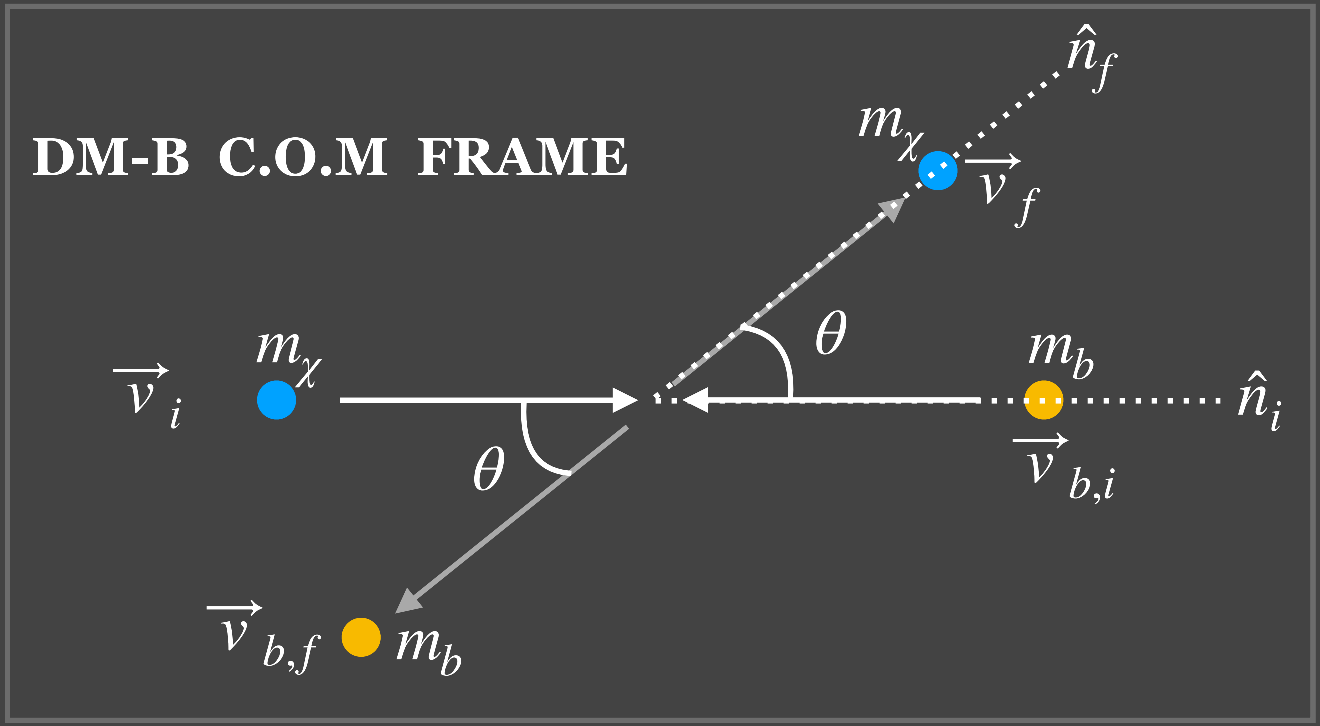
CMB+Lyα $\sigma_0(m_\chi)$ bounds ($n = -4, -2, -1, 0, 2$)



Xu et. al, PRD **97**, 103530, 2018

- Flat profile: $m_\chi \ll m_b$, $R_\chi \propto \sigma_0 (v_{th})^{n+1}/M \propto \sim \sigma_0 v_b^{n+1}$ down to $m_\chi \sim 10MeV$ bec $T_\chi/T_b \ll m_\chi/m_b$ still
  - m_χ dep. drops out
- $\sigma_0 \propto m_\chi$ profile: For $m_\chi \gg m_b$, $R_\chi \propto \sigma_0 (v_{th})^{n+1}/M \propto \sim \sigma_0 v_b^{n+1}/M$
  - $R_\chi \propto \sigma_0/m_\chi$

DM-baryon (χ-b) Scattering



Relevant quantities (for ex.):

1. Rate of scattering for DM: $\Gamma_{\chi b}\left(f_\chi(\vec{v}), \sigma_{\chi b}\right)$,
   $v_{\chi b} = |\vec{v}_\chi - \vec{v}_b|$
2. DM Heat exchange rate: $\dot{Q}_\chi(f_\chi(\vec{v}), \sigma_{\chi b}) \propto \dot{T}_\chi^{scatt}$
3. DM momentum exchange rate: $\dot{\vec{p}}_\chi(f_\chi(\vec{v}), \sigma_{\chi b}) = m_\chi \dot{\vec{V}}_\chi$