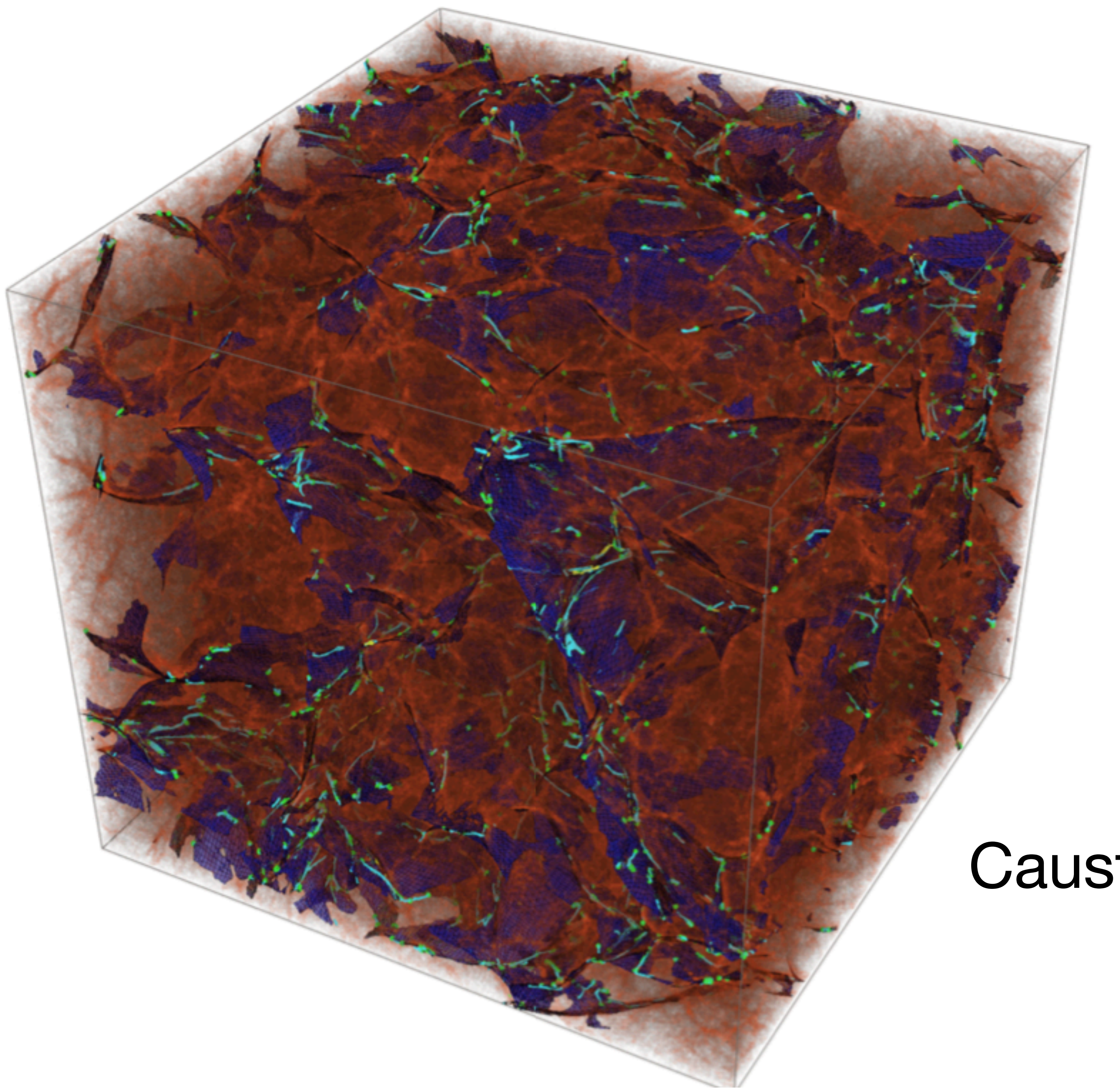
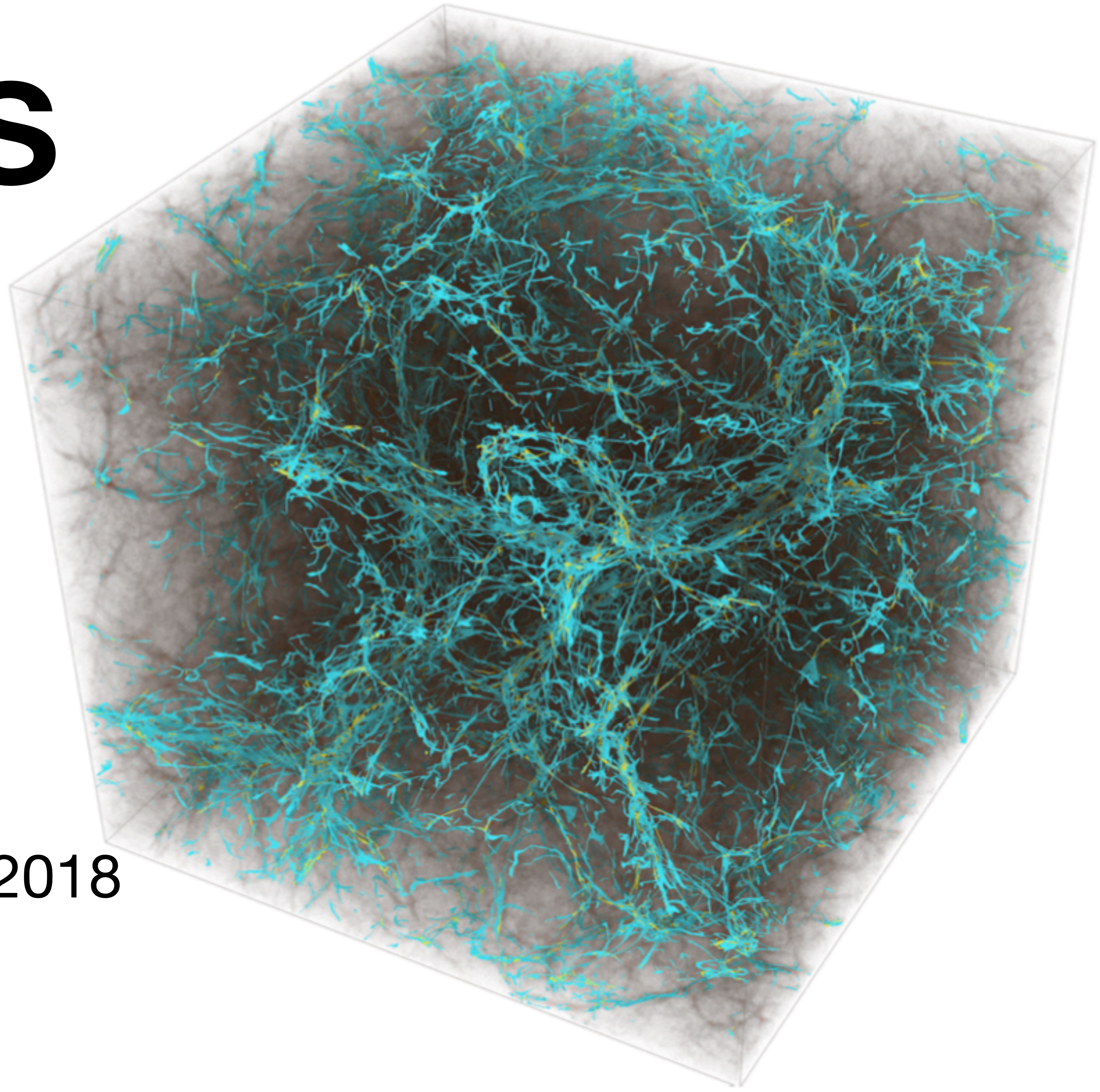
Dissecting the cosmic web with caustics



Job Feldbrugge
University of Edinburgh
Carnegie Mellon University

Caustic Skelton & Cosmic Web, JCAP, Issue 5, 2018



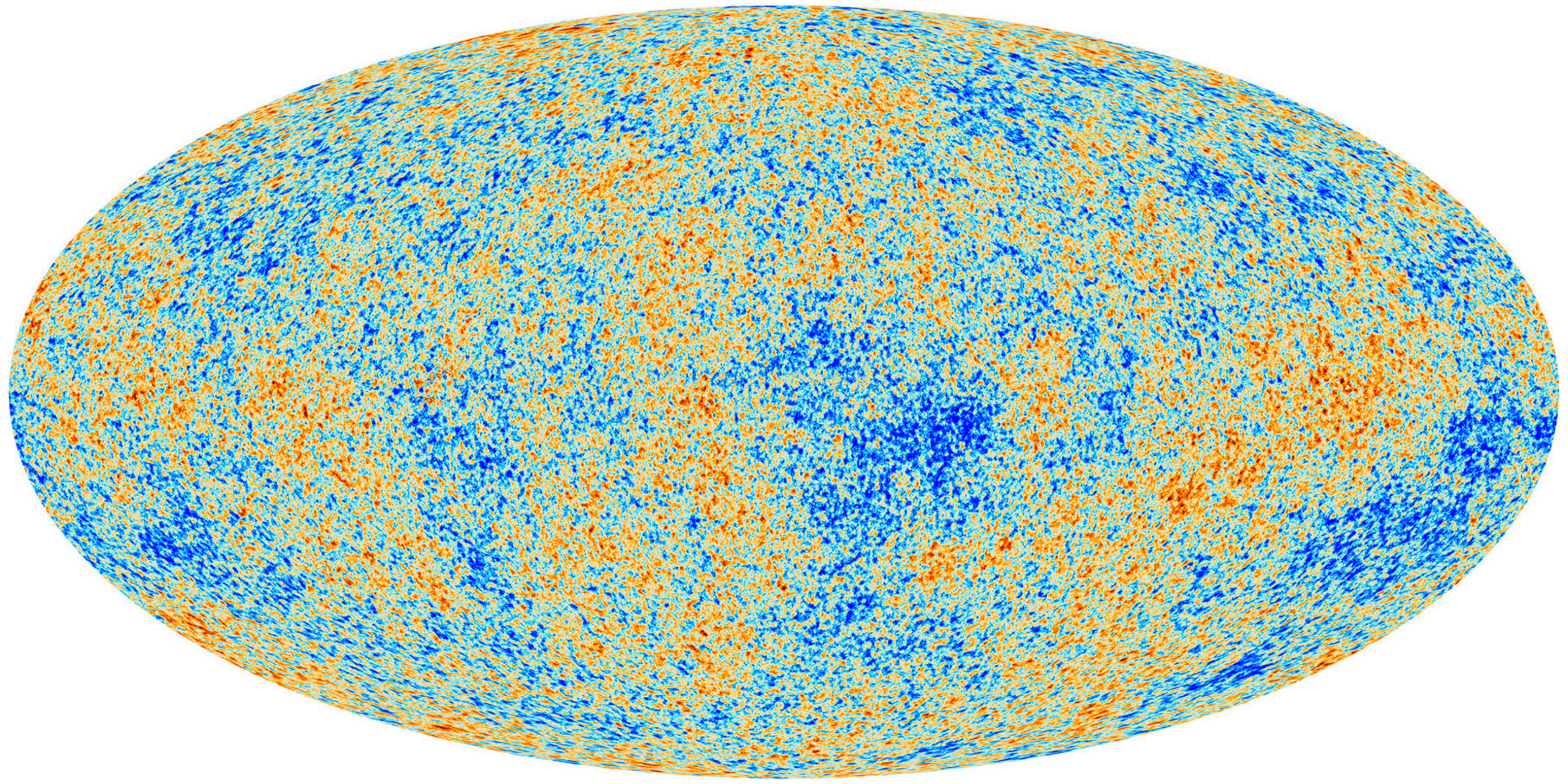
Cosmology from home 2022

Collaborators: R. van de Weygaert, J. Hidding, G. Wilding, K. Bixerman, Y. Yan, Y. Sklansky, and Joost Feldbrugge

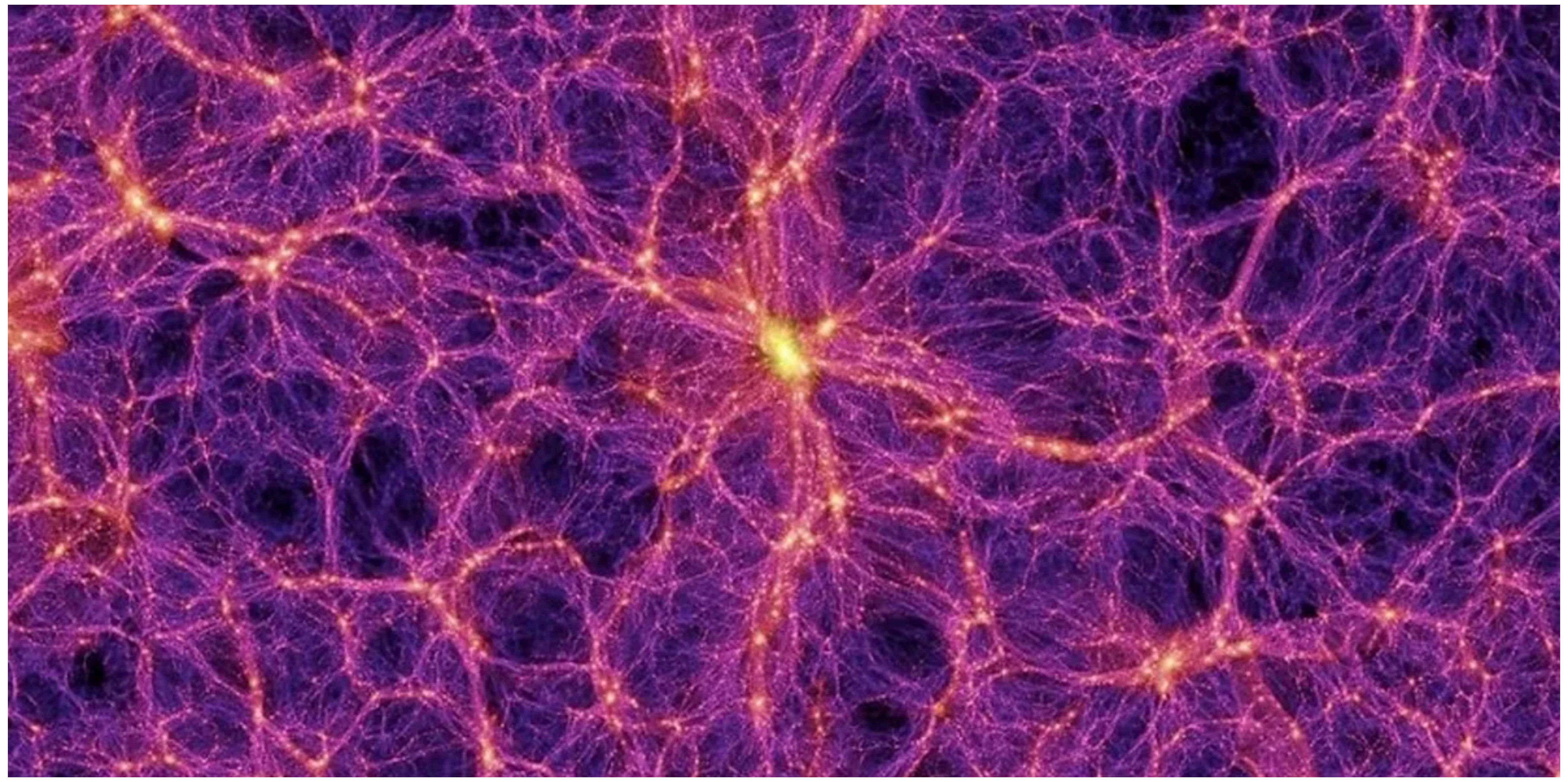
Late time cosmology

- The Universe originated in the hot big bang. The tiny (quantum) density fluctuations in the plasma are extremely simple: close to a Gaussian random field.

- After 380000 years at 3000 Kelvin, these fluctuations grew into the *cosmic web*, consisting of **voids**, **walls**, **filaments**, and **clusters** under non-linear gravitational collapse.



Planck Satellite



Millenium simulation

Lagrangian fluid dynamics

- Lagrangian fluids

$$\mathbf{x}(t) = \mathbf{q} + \mathbf{s}_t(\mathbf{q})$$

- Zel'dovich approximation (1970):

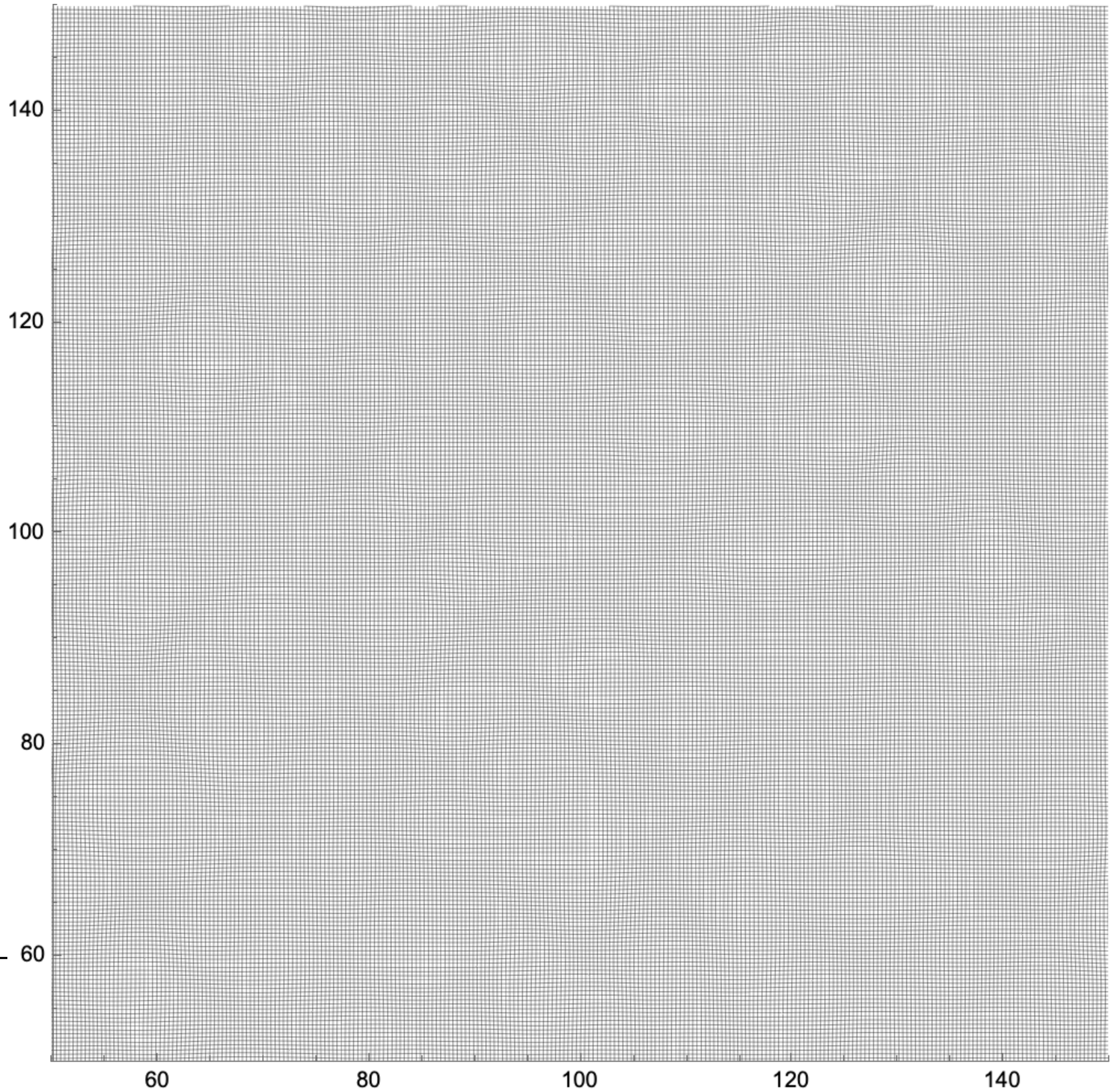
$$\mathbf{s}_t(\mathbf{q}) = -D_+(t)\nabla\Psi(\mathbf{q})$$

$$\nabla^2\Psi(\mathbf{q}) = 4\pi G\rho_0\delta(\mathbf{q})$$

- Shell-crossing: density spikes at the caustics

$$\mathcal{M}(\mathbf{q}) = \nabla\mathbf{s}(\mathbf{q}) \quad \mathcal{M}(\mathbf{q})\mathbf{v}_i(\mathbf{q}) = \mu_i(\mathbf{q})\mathbf{v}_i(\mathbf{q})$$

$$\rho_t(\mathbf{x}') = \sum_{\mathbf{q}\in\mathbf{x}_t^{-1}(\mathbf{x}')} \frac{\rho_i(\mathbf{q})}{|1+\mu_1(\mathbf{q},t)||1+\mu_2(\mathbf{q},t)|}$$



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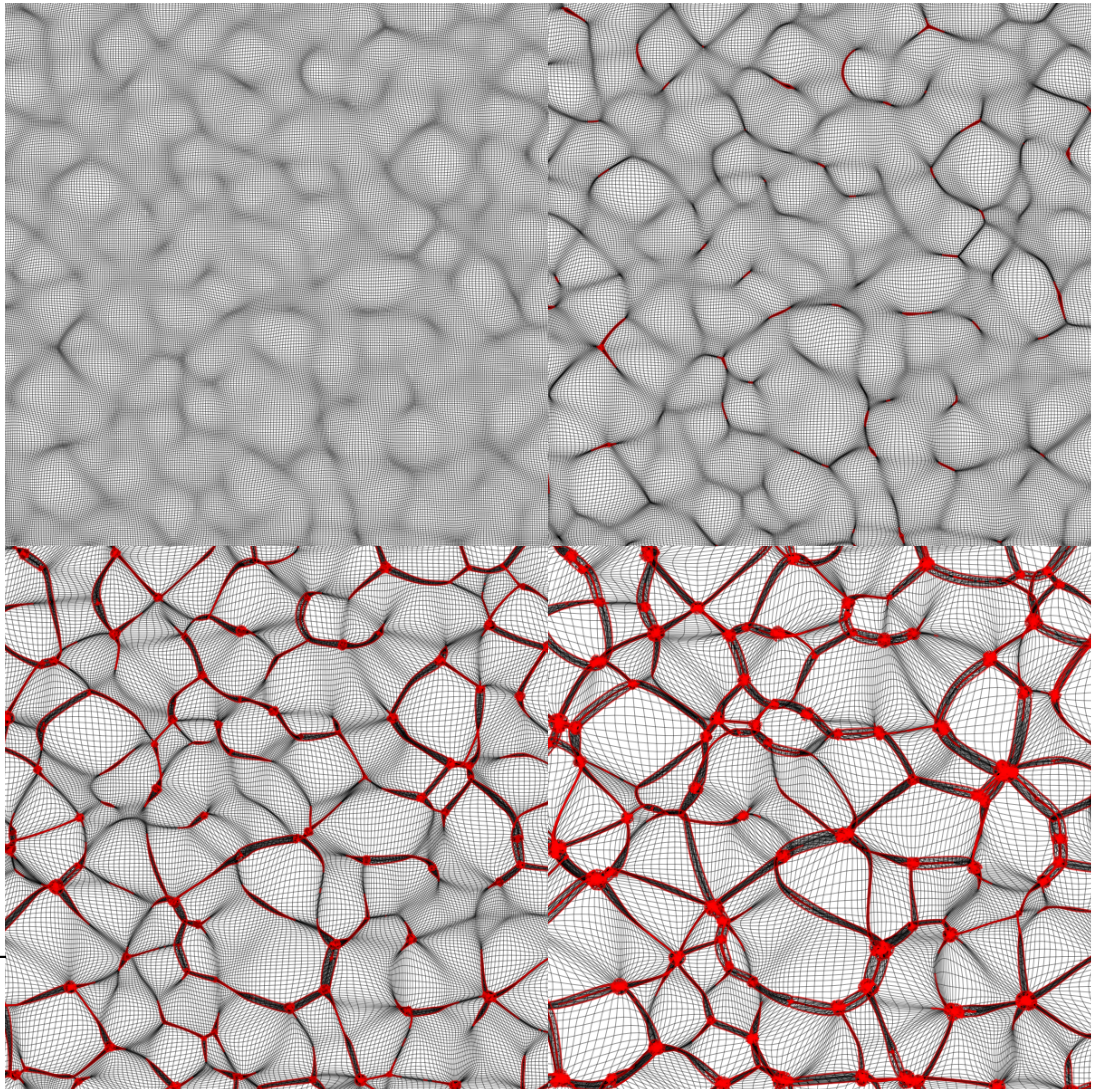


Caustics and Catastrophes

- *Vladimir Arnol'd* extended *René Thom* classification of stable degenerate critical points to **Lagrangian catastrophe theory**
- The **classification of caustics** was applied to *large-scale structure formation* to predict the geometric structure of the *cosmic web*
- Renewed interest, Hidding et al (2013)

1972 NORMAL FORMS FOR FUNCTIONS NEAR DEGENERATE CRITICAL POINTS, THE WEYL GROUPS OF A_k, D_k, E_k AND LAGRANGIAN SINGULARITIES

V. I. Arnol'd

1980 EVOLUTION OF SINGULARITIES OF POTENTIAL FLOWS IN COLLISION-FREE MEDIA AND THE METAMORPHOSIS OF CAUSTICS IN THREE-DIMENSIONAL SPACE

V. I. Arnol'd

1982 The Large Scale Structure of the Universe I. General Properties. One- and Two-Dimensional Models

V. I. ARNOLD
Moscow State University, U.S.S.R.

and

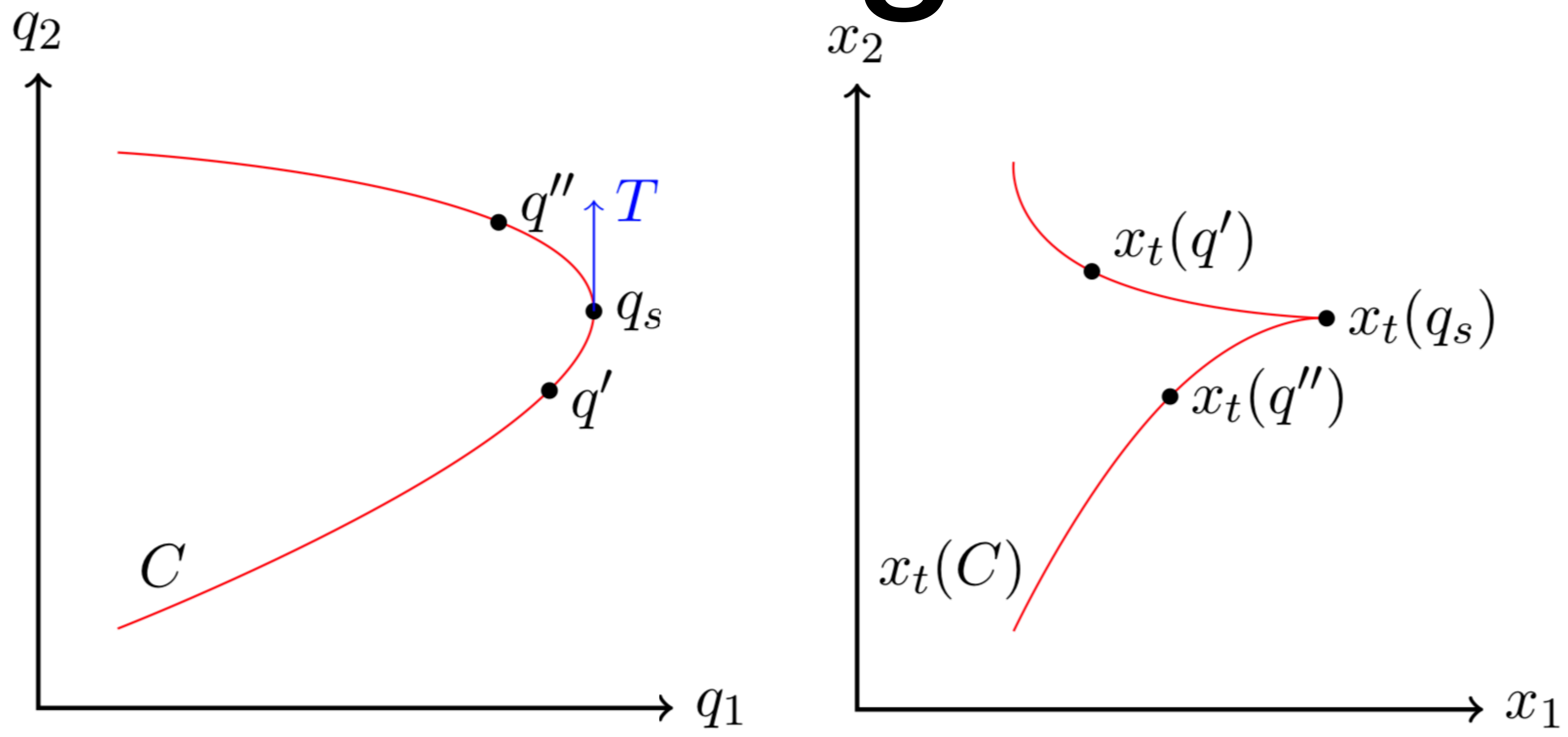
S. F. SHANDARIN and YA. B. ZELDOVICH
Institute of Applied Mathematics, Moscow, U.S.S.R.

The caustic skeleton in 3D

Singularity class	Singularity name	Feature in the 2D cosmic web	Feature in the 3D cosmic web
A_2	fold	collapsed region	collapsed region
A_3	cusp	filament	wall or membrane
A_4	swallowtail	cluster or knot	filament
A_5	butterfly	not stable	cluster or knot
D_4	hyperbolic/elliptic	cluster or knot	filament
D_5	parabolic	not stable	cluster or knot

The identification of the different caustics in the 2- and 3-dimensional cosmic web

Shell-crossing condition



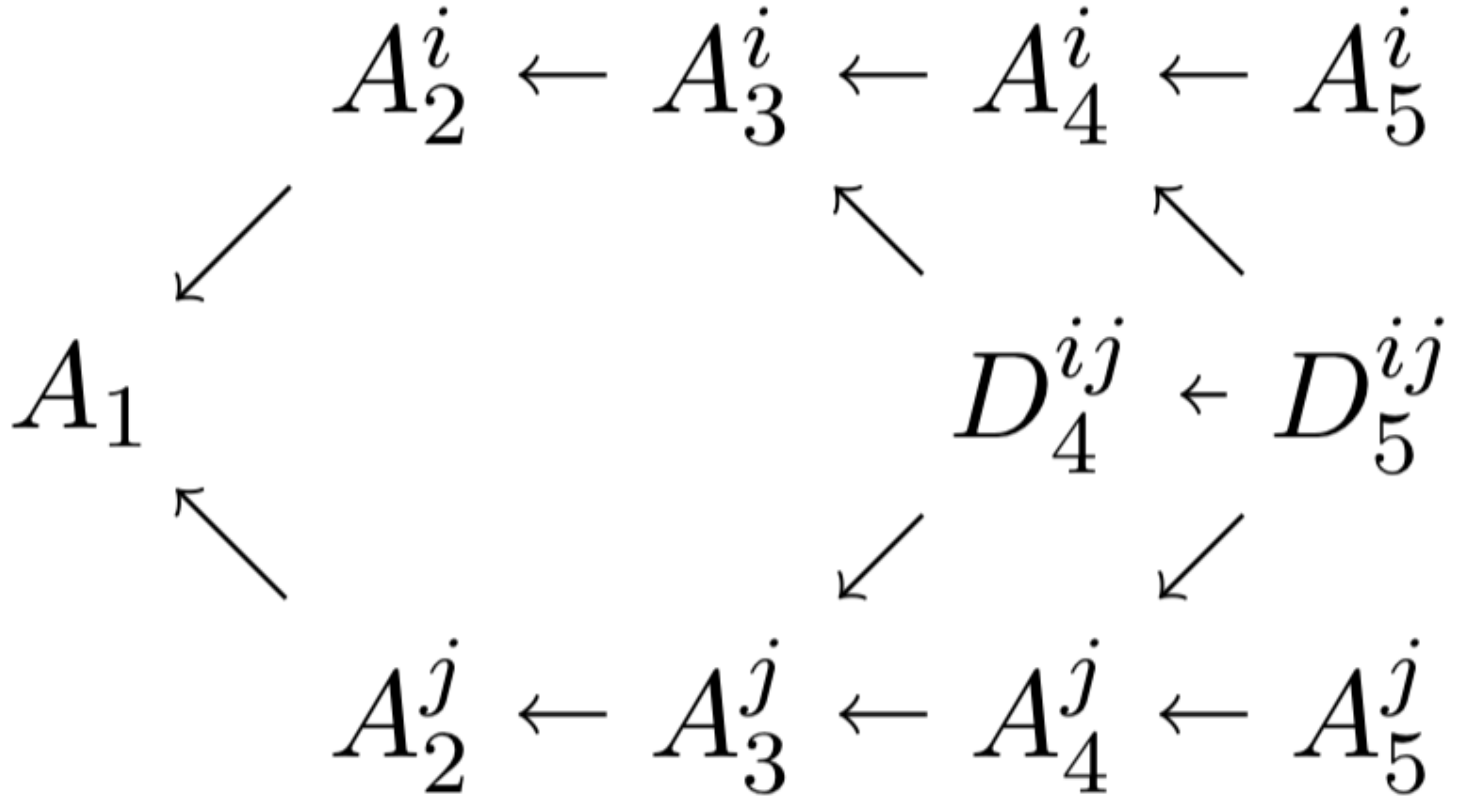
Theorem: *A manifold $M \subset L$ forms a singularity under the mapping x_t in the point $x_t(q_s) \in x_t(M) \subset E$ at time t, meaning that $x_t(M)$ is not smooth in $x_t(q_s)$, if and only if there exists at least one nonzero tangent vector $T \in T_{q_s}M$ satisfying*

$$(1 + \mu_{it}(q_s))v^*_{it}(q_s) \cdot T = 0$$

for all $i = 1, 2, \ldots, dim(L)$.

Feldbrugge et al (2019)

Caustic conditions



Iterative application of the shell-crossing condition

$$(1+\mu_{it}(q_s))v_{it}^*(q_s)\cdot T = 0$$

leads to the caustic conditions on both the eigenvalue and eigenvector fields:

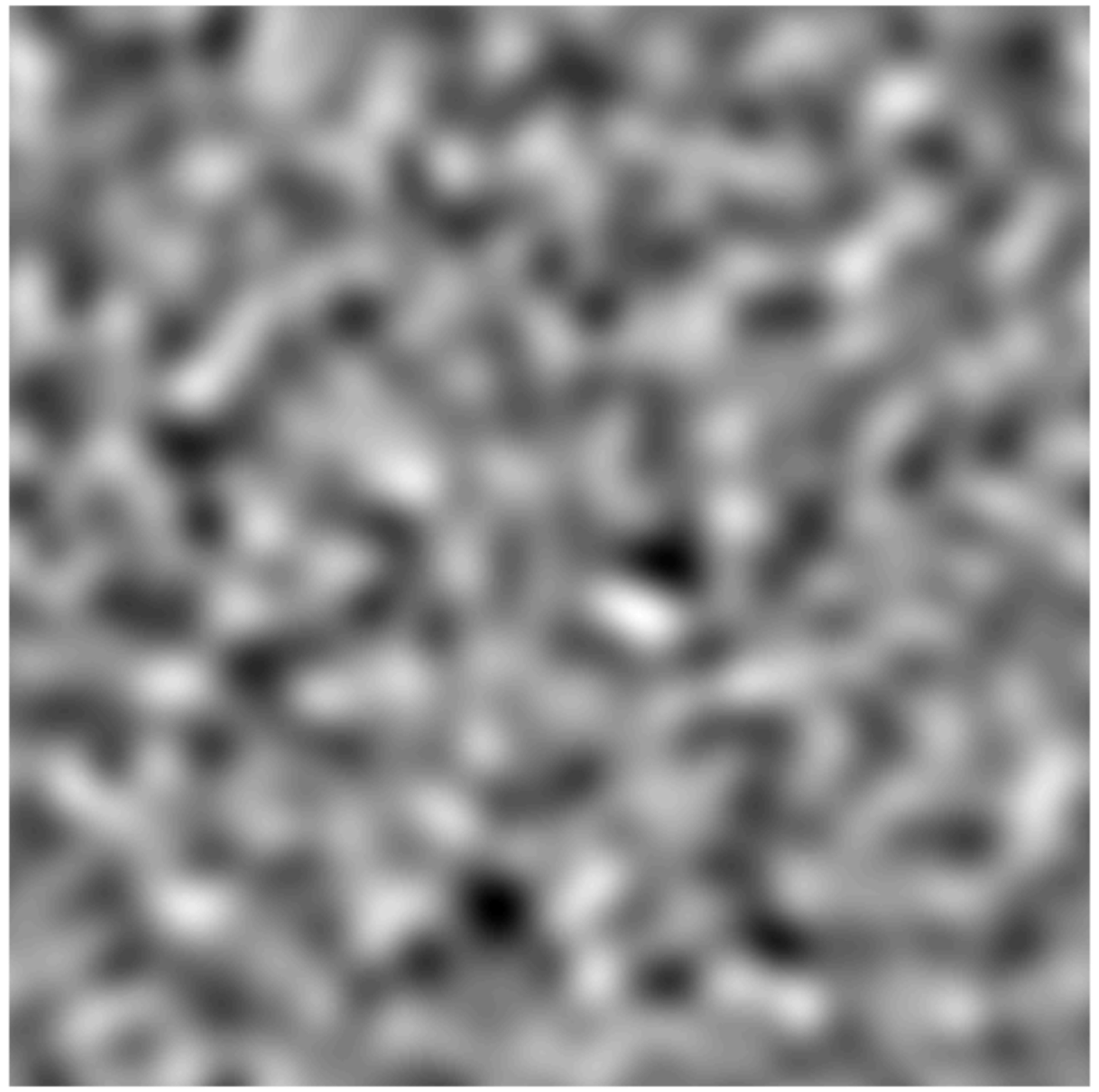
Fold:	$A_2^i(t) = \{\mathbf{q} \in L \mid 1+\mu_{it}(\mathbf{q}) = 0\}$
Cusp:	$A_3^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_2^i(t), \mathbf{v}_i \cdot \nabla\mu_{it} = 0\}$
Swallowtail:	$A_4^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_3^i(t), \mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla\mu_{it}) = 0\}$
Butterfly:	$A_5^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_4^i(t), \mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla(\mathbf{v}_i \cdot \nabla\mu_{it})) = 0\}$
Umbilic:	$D_4^{ij}(t) = \{\mathbf{q} \in L \mid 1+\mu_{it}(\mathbf{q}) = 1+\mu_{jt}(\mathbf{q}) = 0\}$
Parabolic:	$D_5^{ij}(t) = \{\mathbf{q} \in L \mid q \in D_4^{ij}(t), \mathbf{v}_i \cdot \nabla\mu_i = \mathbf{v}_j \cdot \nabla\mu_j = 0\}$

Morse-Smale theory of full deformation tensor field. No free parameters!

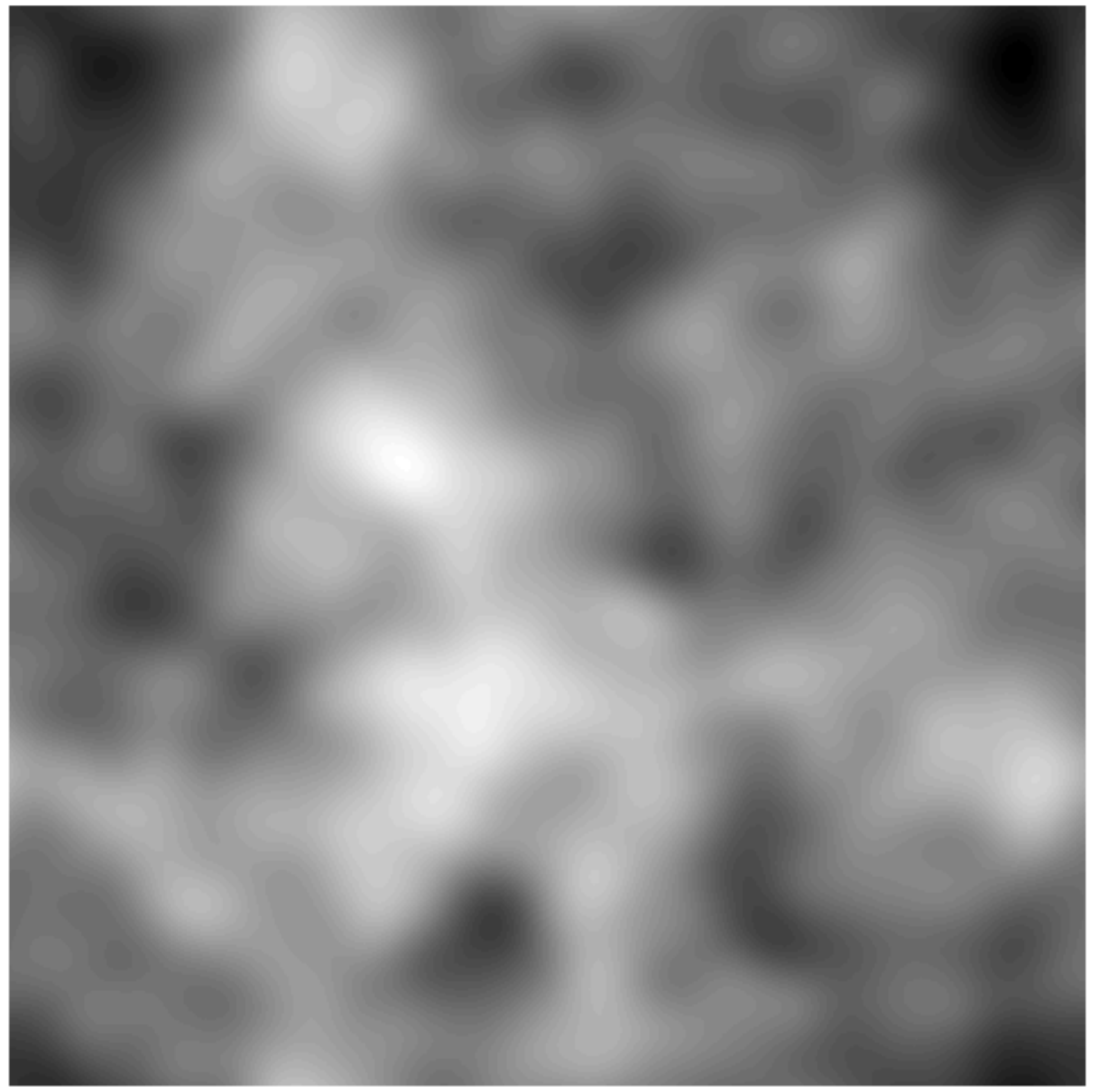
Feldbrugge et al (2019)

Density perturbation v.s. eigenvalue fields

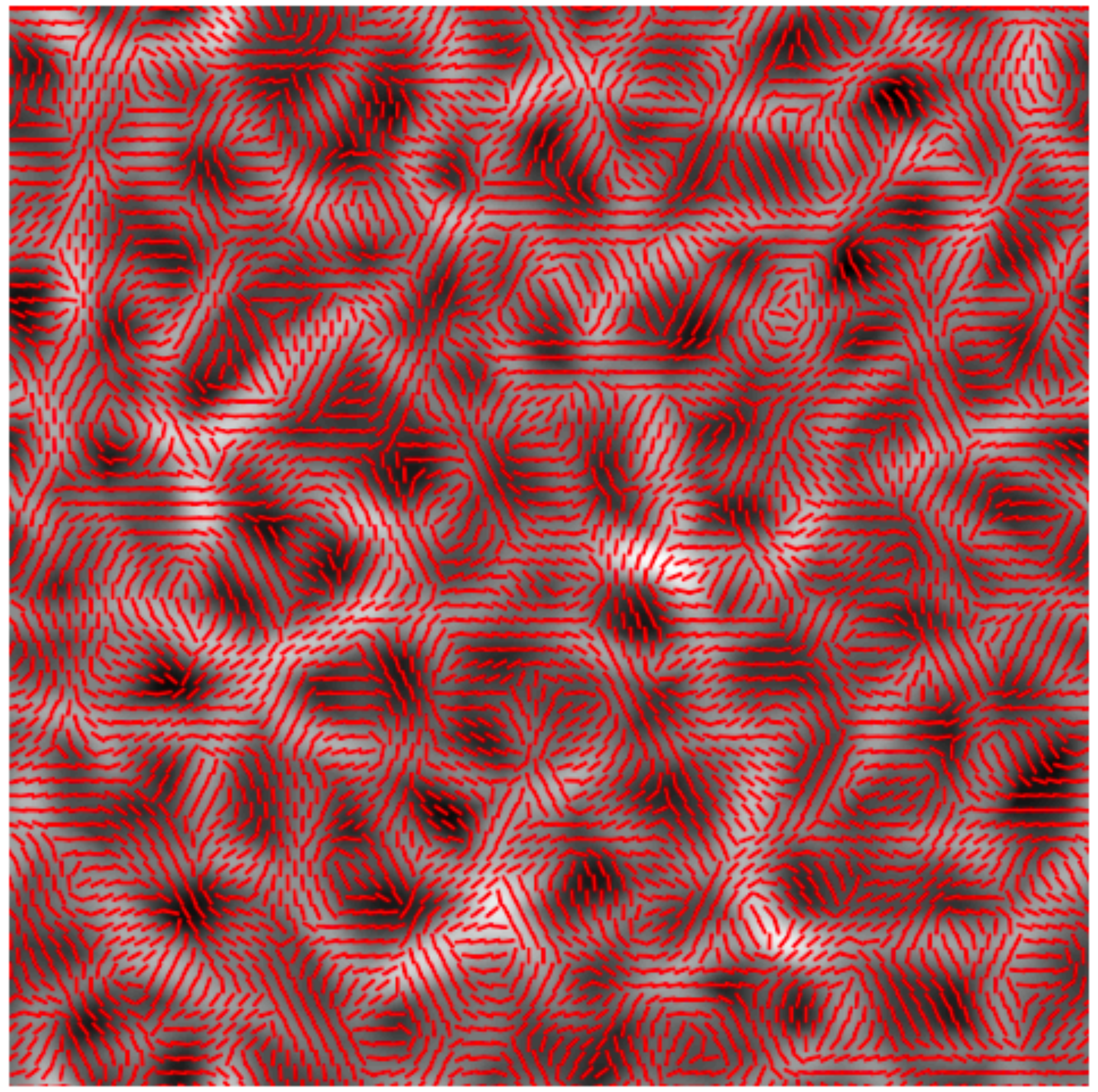
- The eigenvalue and eigenvector fields are non-linear transformations of the density perturbations
- The web-like nature is embedded in the distribution of the eigenvalue and eigenvector fields



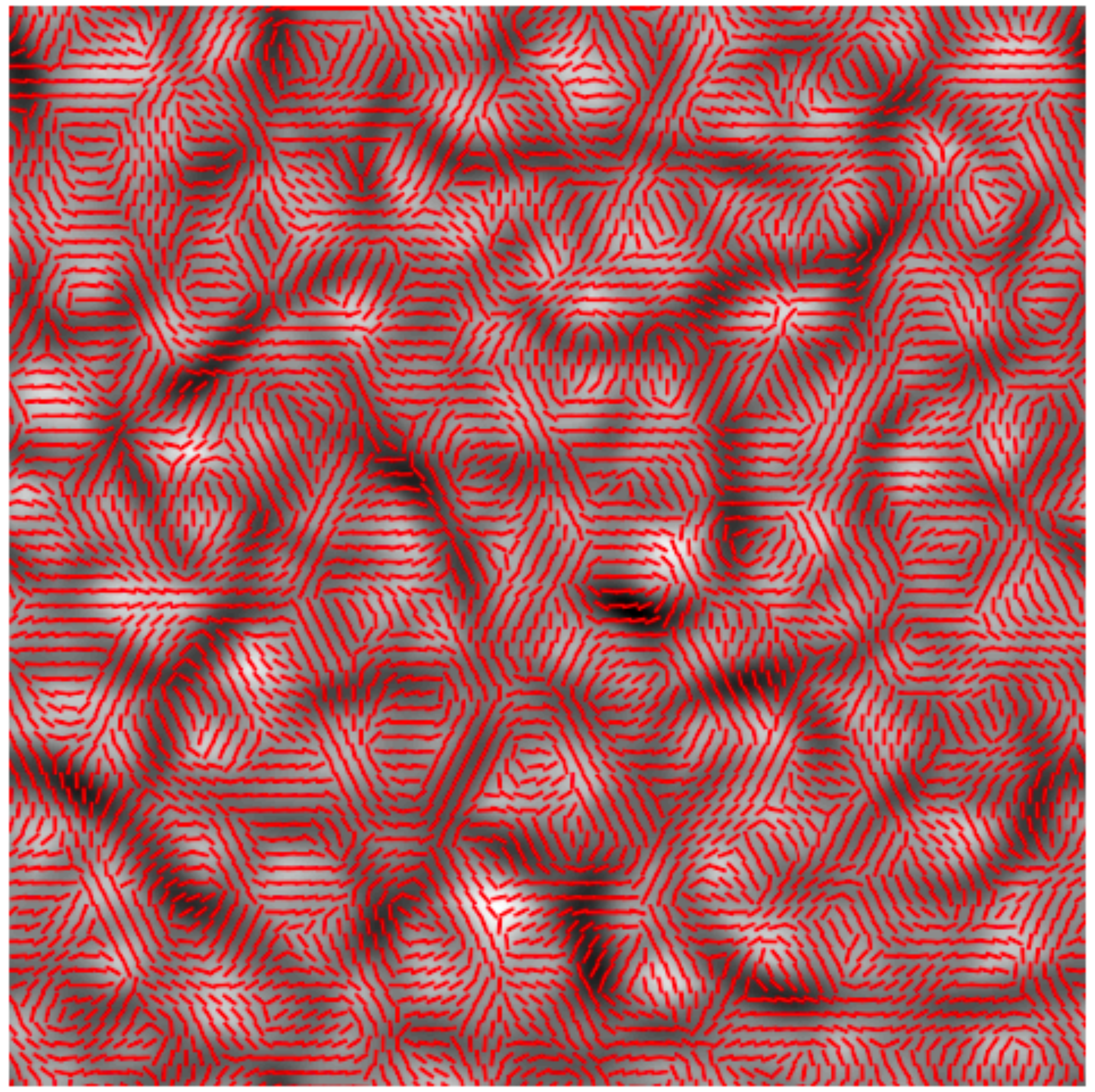
(a) The density perturbation δ



(b) The displacement potential Ψ

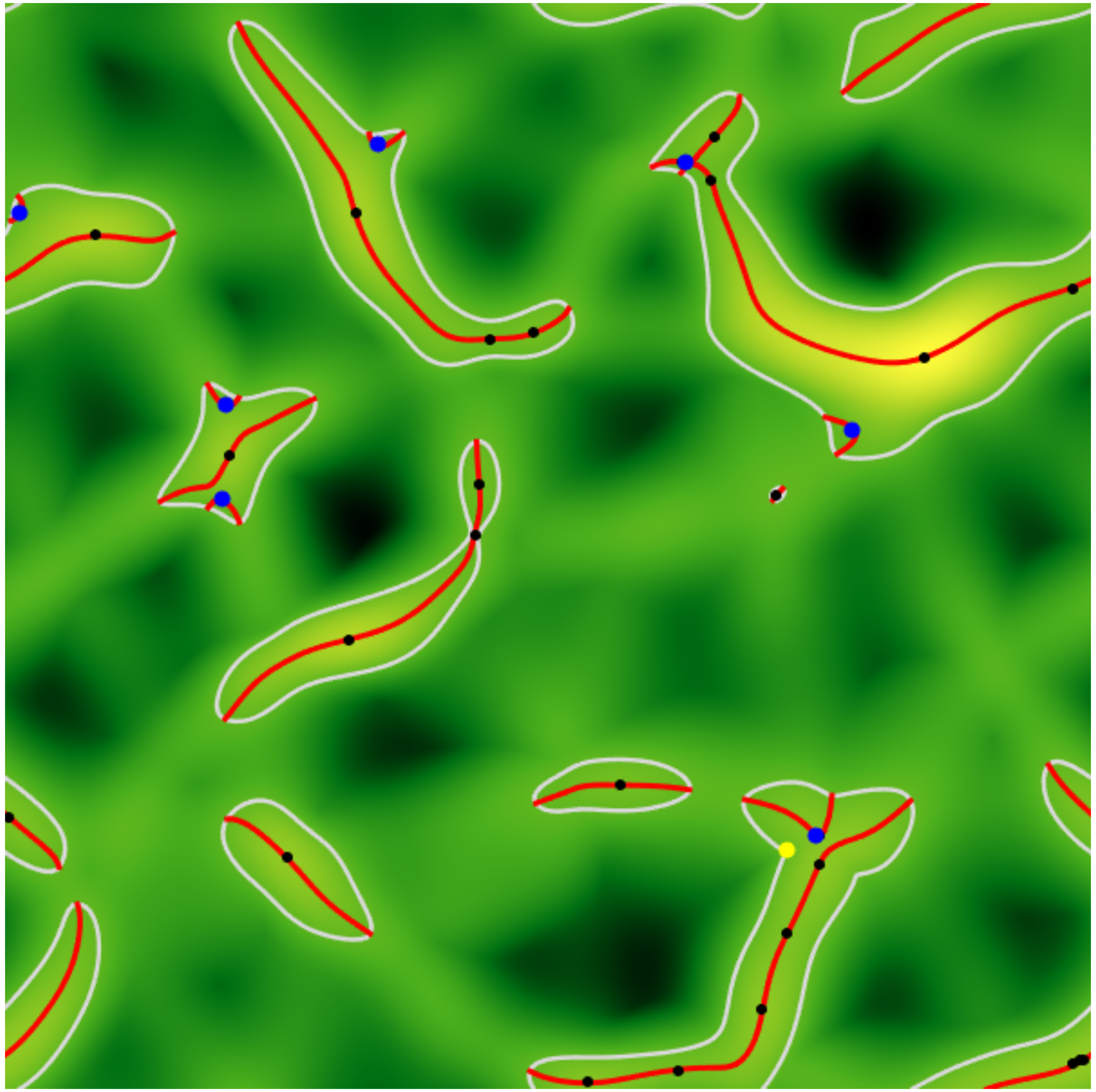


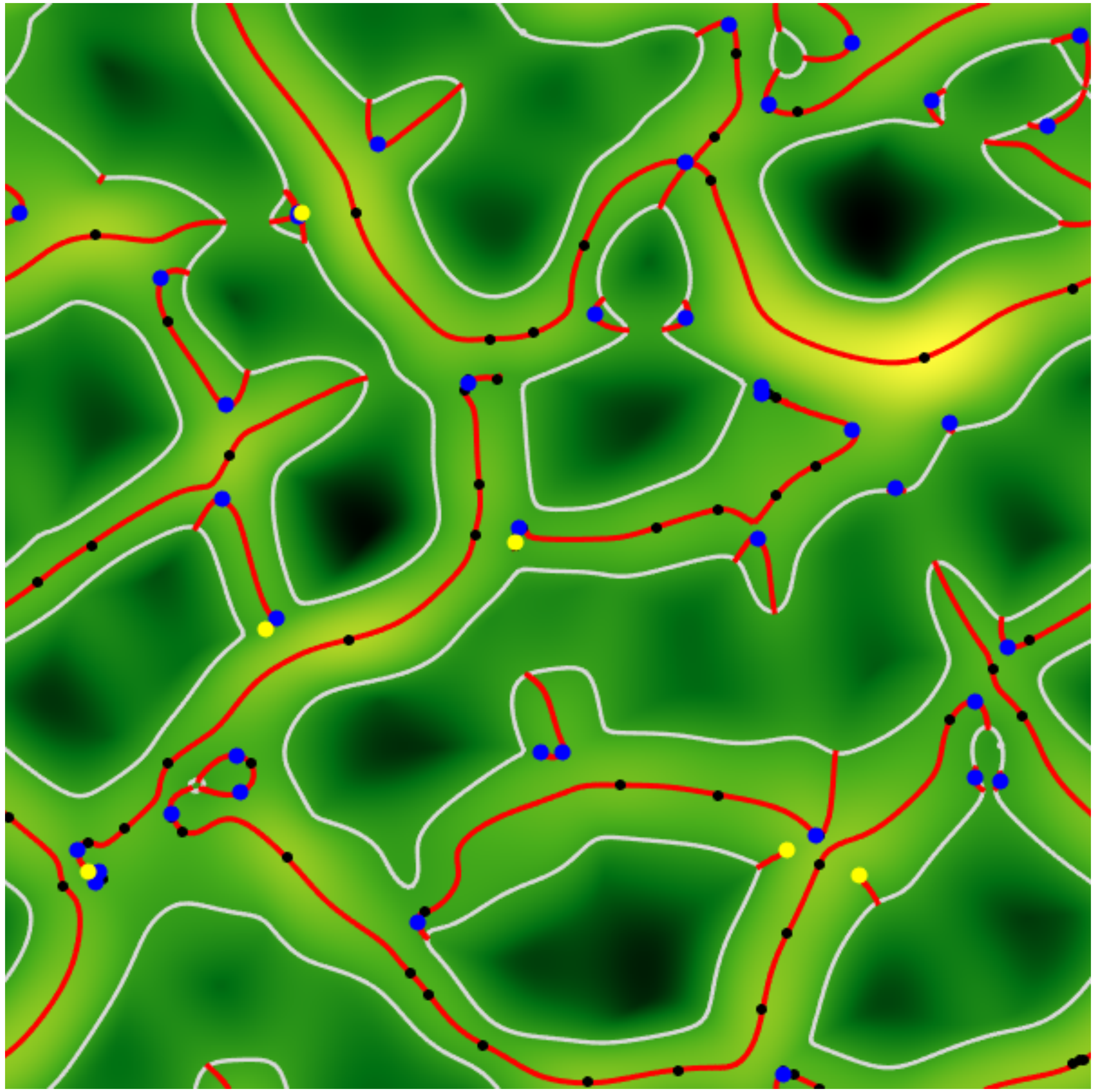
(c) The first eigenvalue and eigenvector fields λ_1, and $\boldsymbol{v}_1$



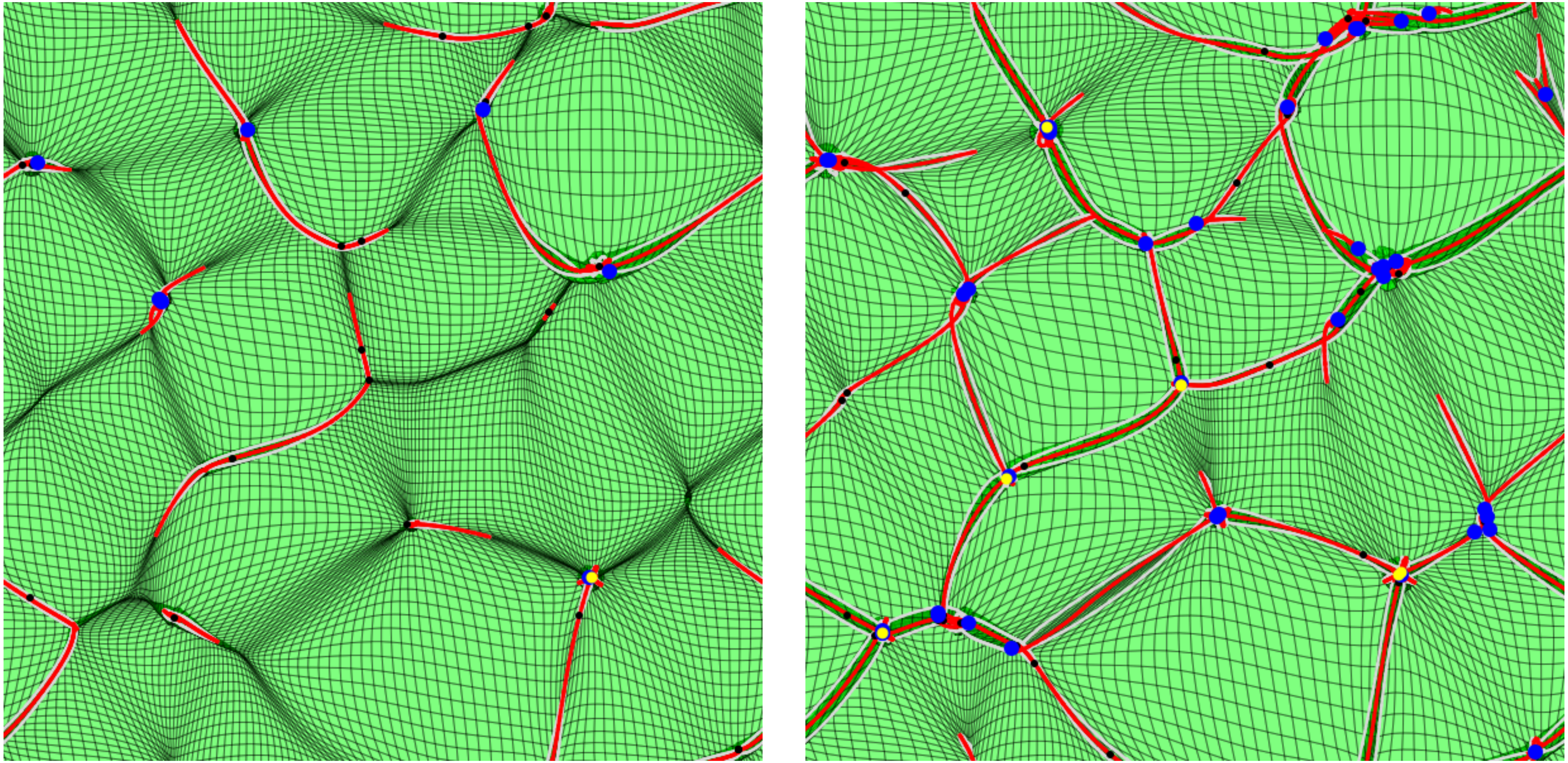
(d) The second eigenvalue and eigenvector fields λ_2, and $\boldsymbol{v}_2$

Caustic skeleton





Caustic skeleton



Skeleton in 3D

Skeleton in 3D

Skeleton in 3D

Dictionary: early and current universe

- The caustic skeleton is a **dictionary** between the **initial conditions** and the geometry/connectivity of **the cosmic web**, based on the underlying dynamics
- Connection between **the simple Gaussian** statistics and **the intricate non-linear universe**
- **Eigenvalue and eigenvectors**

$$
\begin{array}{ll}
\text{Fold:} & A_2^i(t) = \{\mathbf{q} \in L \mid 1 + \mu_{it}(\mathbf{q}) = 0\} \\
\text{Cusp:} & A_3^i(t) = \{\mathbf{q} \in L \mid \mathbf{q} \in A_2^i(t), \mathbf{v}_i \cdot \nabla \mu_{it} = 0\} \\
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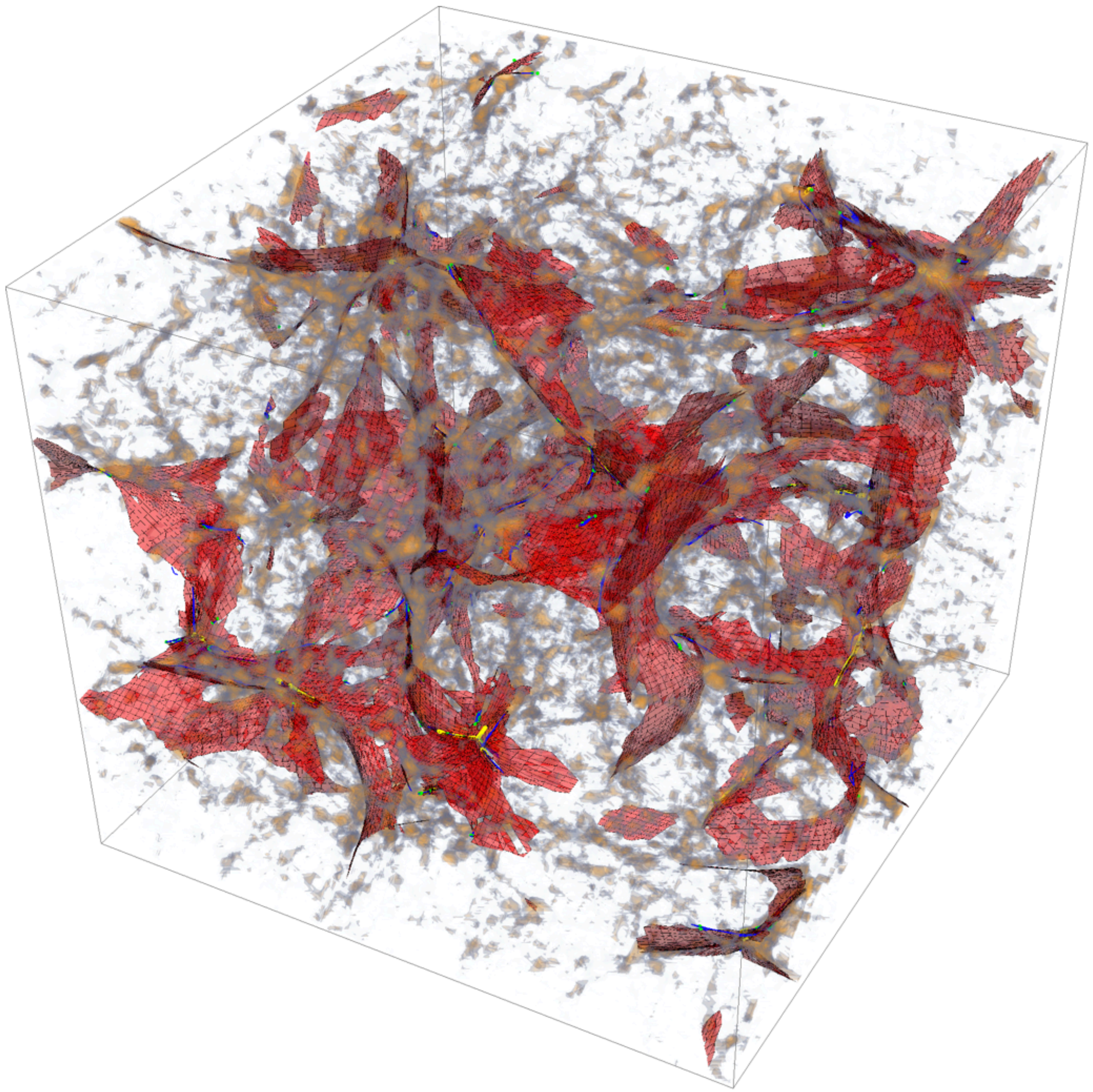
\end{array}
$$

And the corresponding Morse-points on the skeleton

Dressing the caustic skeleton

- Assessing the mass distribution in and around the caustic spine of the cosmic web:
  - What are the properties of walls/filaments/clusters?
  - What is their mass distribution?
  - How do they form and relate to the initial conditions?
  - Detailed merging history and hierarchical evolution of the filamentary network?



Constrained GRF theory

- Generate Gaussian random fields subject to constraints.
- Linear constraints: Hoffman-Ribak algorithm:

Bertschinger 1987
Hoffman, Ribak 1991
van de Weygaert & Bertschinger 1996

  - The statistical properties of the residue of a cGRF with respect to the mean field is independent of the values of the constraints
  - Generate GRF ➔ Measure constraint values ➔ Evaluate corresponding mean field ➔ Evaluate residue ➔ Add residue to mean field with target constraint values
- Algorithm only works for linear constraints on a Gaussian field

Log normal model: Ravi Sheth 1997

Constrained GRF theory

Given the constraints: $\Gamma = \{C_i[f; \boldsymbol{q}_i] = c_i, \ \ i = 1, \ldots, M\} \, ,$

Bertschinger 1987
Hoffman, Ribak 1991
Bertschinger and Weygaert 1996

we obtain the mean field:

$$\begin{aligned} \bar{f}_{\boldsymbol{c}}(\boldsymbol{q}) &= \langle f(\boldsymbol{q}) | \Gamma \rangle \\ &= \bar{f}(\boldsymbol{q}) + \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \xi_{ij}^{-1} (c_j - \bar{C}_j) \end{aligned}$$

and the variance of the residue:

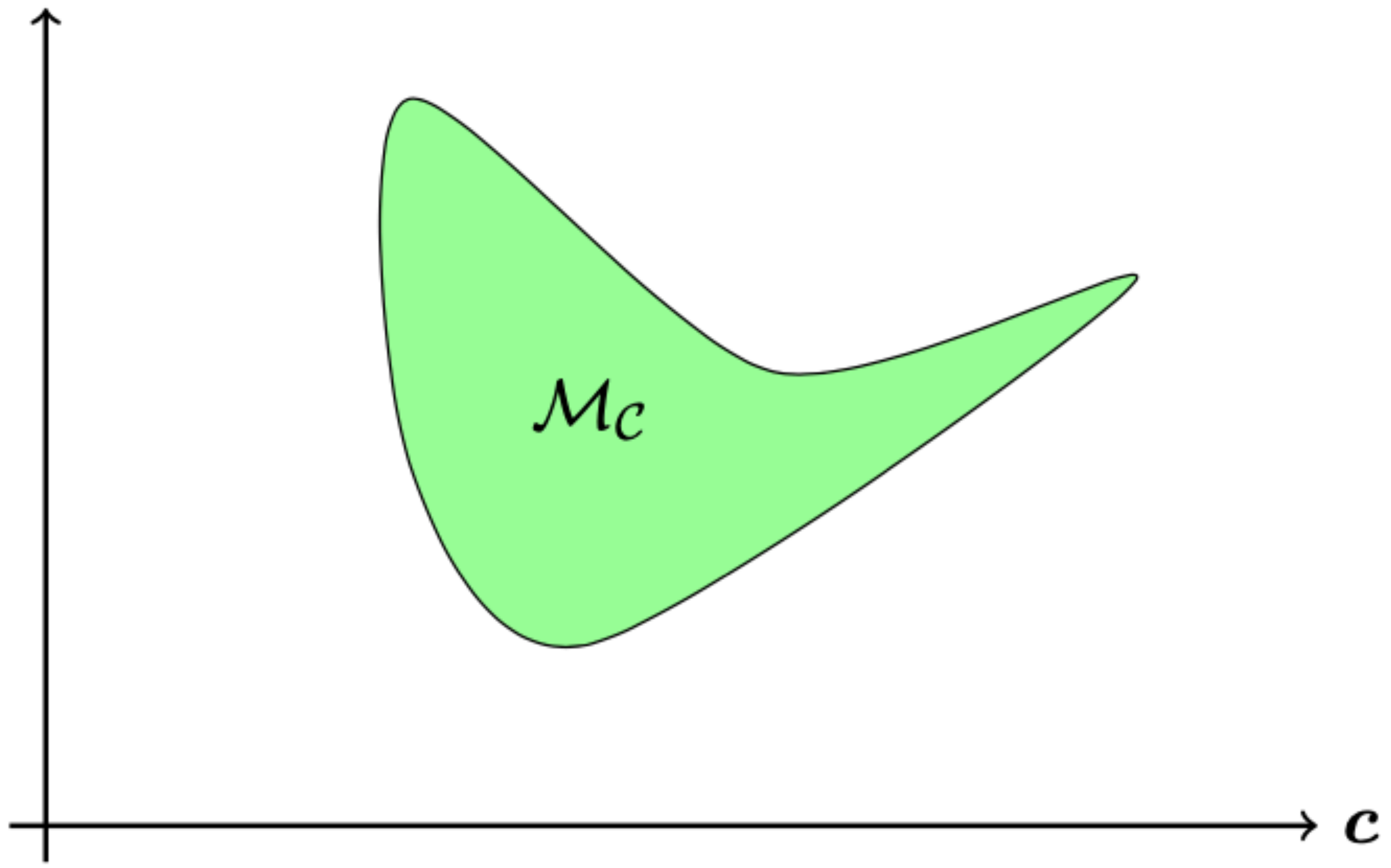
$$\langle \delta f(\boldsymbol{q})^2 | \Gamma \rangle = \sigma_0^2 - \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \xi_{ij}^{-1} \xi_j(\boldsymbol{q})$$

To generate realizations, we generate an unconstrained GRF, find the c_i's, find the residue and add the residue to the mean field of the target constraint values

Non-linear constraints

- The eigenvalue and eigenvector fields are not Gaussian, and the caustic conditions are non-linear. For this reason, we develop non-linear constraint Gaussian random field theory

$$\mathcal{M}_\mathcal{C} = \{\boldsymbol{c} \,|\, \mathcal{C}_i(\boldsymbol{c}) = 0 \text{ for all } i = 1, \ldots, N\}.$$



- On this constraint manifold, we find the induced probability density

$$p(\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathcal{M}_\mathcal{C}) = \frac{p(\boldsymbol{c})}{\int_{\mathcal{M}_\mathcal{C}} p(\boldsymbol{c}) \, \mathrm{d}\boldsymbol{c}}.$$

Non-linear constraints

- Using the properties of the constraint manifold we can leverage the Hoffman-Ribak principle for non-linear constraints:
  - The mean field: $\bar{f}_{\mathcal{C}}(\boldsymbol{q}) = \bar{f}_{\bar{\boldsymbol{c}}}(\boldsymbol{q})\,, \quad \bar{\boldsymbol{c}} \equiv \langle \boldsymbol{c} \,|\, \Gamma \rangle = \int_{\boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}} \boldsymbol{c}\, p(\boldsymbol{c} \,|\, \boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}) \mathrm{d}\boldsymbol{c}\,,$
  - The variance:

$$\left\langle \delta f(\boldsymbol{q})^2 \,|\, \Gamma \right\rangle = \sigma_0^2 - \sum_{i,j=1}^{M} \xi_i(\boldsymbol{q}) \zeta_{ij}^{-1} \xi_j(\boldsymbol{q})\,, \quad \zeta_{ij}^{-1} = \xi_{ij}^{-1} - \sum_{k,l=1}^{M} \xi_{ik}^{-1} \mathrm{cov}(c_k, c_l \,|\, \boldsymbol{c} \in \mathcal{M}_{\mathcal{C}}) \xi_{lj}^{-1}$$

- To generate realizations, we first sample the constraint manifold. Given the constraint values, we use the Hoffman-Ribak algorithm

- Very efficient and does not require expensive Bayesian inference MCMC techniques

Generating the cusps/filaments

Caustic conditions of the cusp

$$b_+(t)\lambda_1(\boldsymbol{q}_c) = 1\,,$$
$$\boldsymbol{v}_1(\boldsymbol{q}_c)\cdot\nabla\lambda_1(\boldsymbol{q}_c) = 0\,.$$

with the orientation with respect to the eigenvector field

$$\alpha = \operatorname{sign}(\boldsymbol{v}_2\cdot\boldsymbol{n})\arccos\left(\frac{|\boldsymbol{v}_1\cdot\boldsymbol{n}|}{\|\boldsymbol{n}\|}\right) \qquad \boldsymbol{n} = \nabla(\boldsymbol{v}_1(\boldsymbol{q}_c)\cdot\nabla\lambda_1(\boldsymbol{q}_c))\,,$$

In the eigenvector frame, we can express the derivatives of the eigenvalues in terms of the Gaussian random field

$$\boldsymbol{v}_1\cdot\nabla\lambda_1 = \pm T_{111}\,,\quad \boldsymbol{v}_2\cdot\nabla\lambda_1 = \pm T_{112}\,,$$
$$\boldsymbol{v}_1\cdot\nabla\lambda_2 = \pm T_{122}\,,\quad \boldsymbol{v}_2\cdot\nabla\lambda_2 = \pm T_{222}\,.$$

$$\boldsymbol{v}_1\cdot\nabla(\boldsymbol{v}_1\cdot\nabla\lambda_1) = T_{1111} + \frac{3T_{112}^2}{T_{11}-T_{22}}\,,$$

Generating the cusps/filaments

Generating the umbilics/clusters

Caustic condition

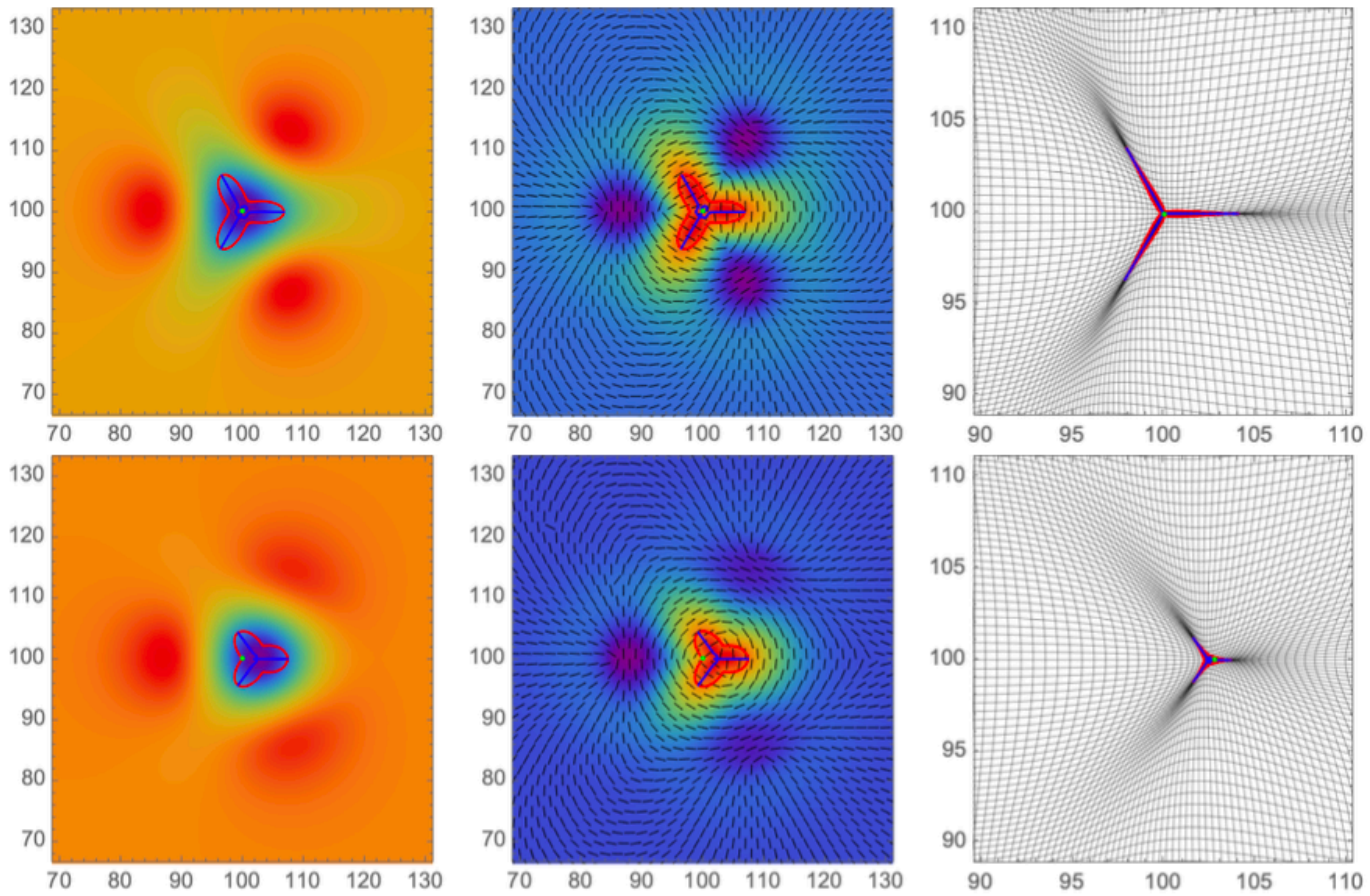
$$b_+(t)\lambda_1(\boldsymbol{q}_c) = b_+(t)\lambda_2(\boldsymbol{q}_c) = 1\,.$$

Elliptic umbilic:

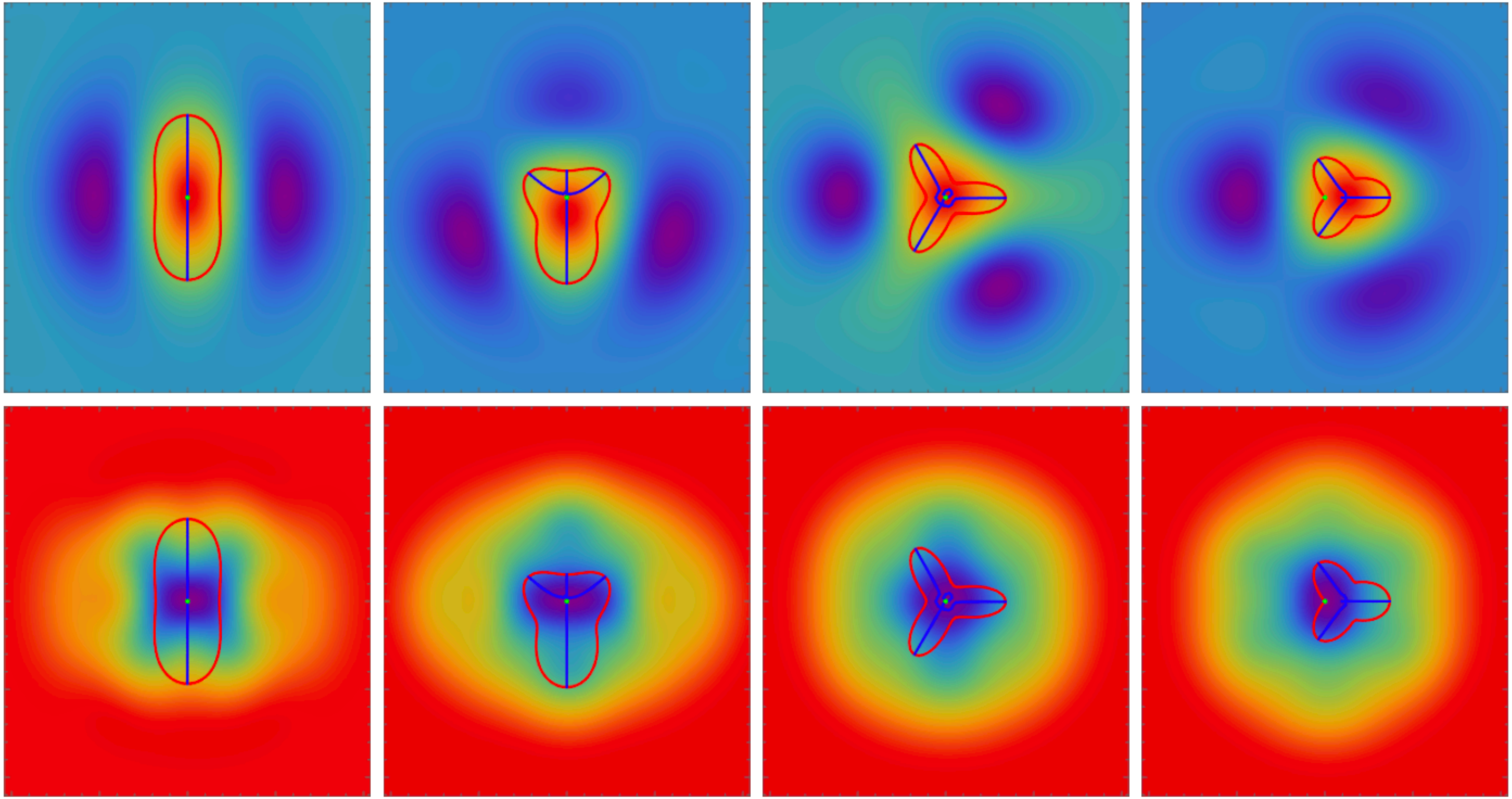
$$\det(\mathcal{H}\,[\det(I - b_+\boldsymbol{\psi})]) > 0\,.$$

Hyperbolic umbilic:

$$\det(\mathcal{H}\,[\det(I - b_+\boldsymbol{\psi})]) < 0\,.$$



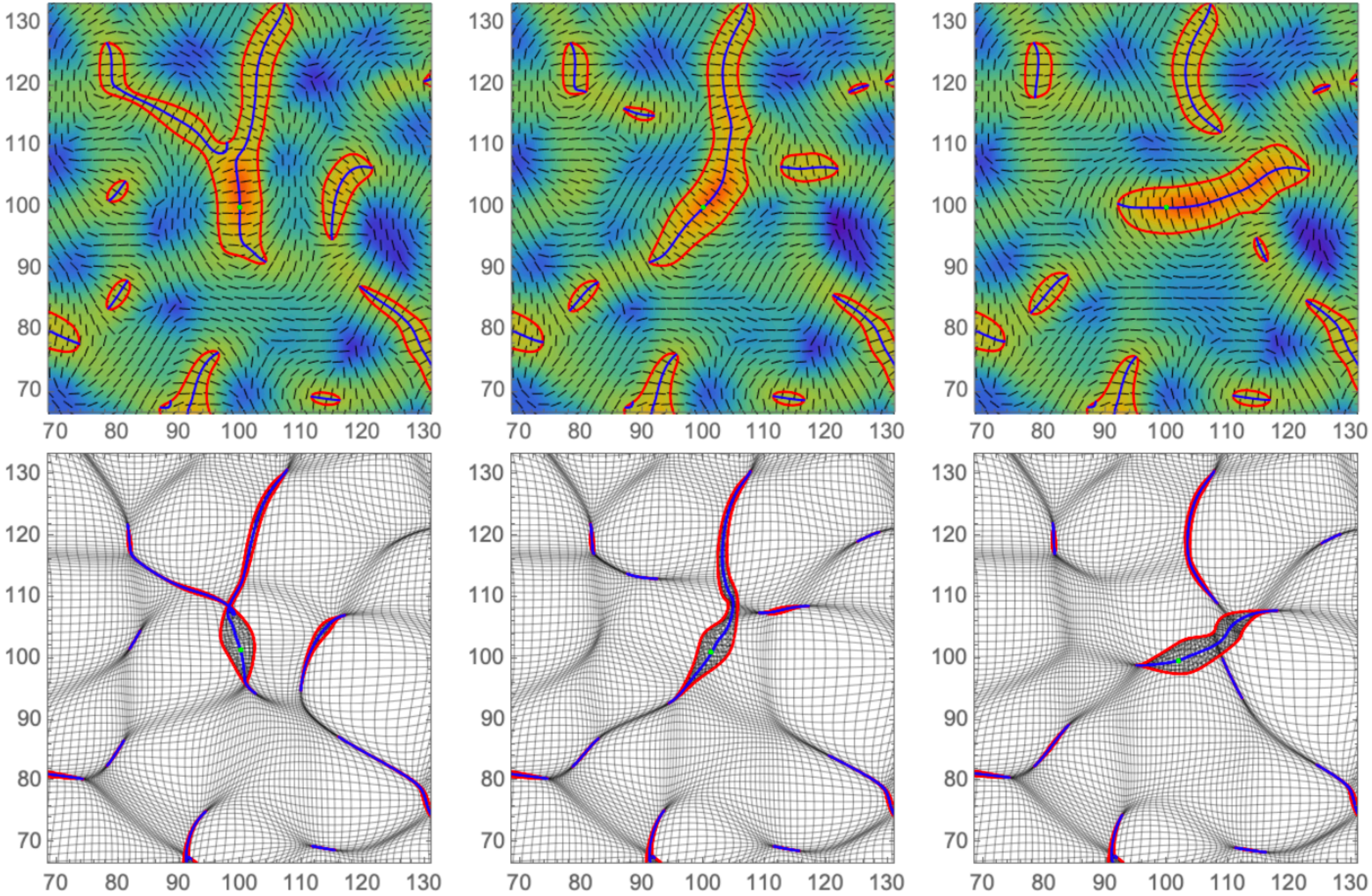
The statistics of the residue



Realizations

Realizations

We can locally change the orientation of the filament



Summary

- **Shell-crossing condition** enables us to derive **caustic conditions in 3D**
- **Caustic skeleton of cosmic web** depends on the **eigenvalue** and **eigenvector fields**
- **Filaments and walls do not require multiple shell-crossings**
- **We extend constrained Gaussian random field theory to non-linear constraints**
- **We generate constrained initial conditions tied to the dynamics of structure formation**
- **Using these techniques we can dress the caustic skeleton and systematically study the cosmic web**
- **Combining with ML, we want to study data**