Perturbation theory in Inflation and beyond

Giovanni Tambalo
AEI Potsdam

With M. Celoria, P. Creminelli and V. Yingcharoenrat

[arXiv:2103.09244 +]

Cosmology From Home 2021

Outline

- Perturbation theory (PT) in Inflation
- Why going beyond PT
- Example in Quantum Mechanics
- The case of Inflation
- Conclusions & future directions

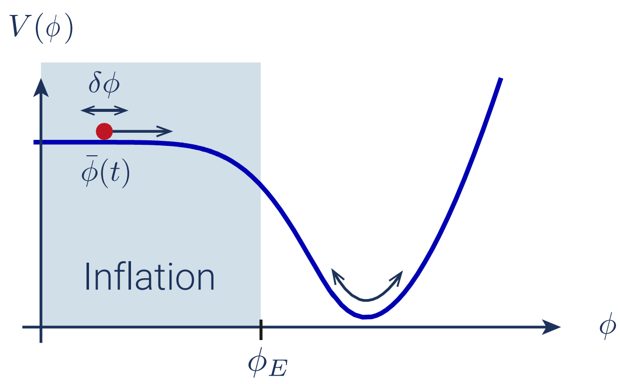
Slow-roll Inflation

- Inflation: period of early acceleration

- Inflaton ϕ rolls down its potential.

  Approximate de Sitter expansion:

$$ds^2 = \frac{-d\eta^2 + d\boldsymbol{x}^2}{\eta^2}$$



- Curvature perturbations ζ freeze outside of the horizon for $\hbar \neq 0$

$$h_{ij} = a^2 \left[e^{2\zeta}\delta_{ij} + \gamma_{ij} \right] \ , \qquad \langle \zeta_{\vec{k}} \zeta_{-\vec{k}} \rangle' = \frac{P_\zeta}{k^3}$$

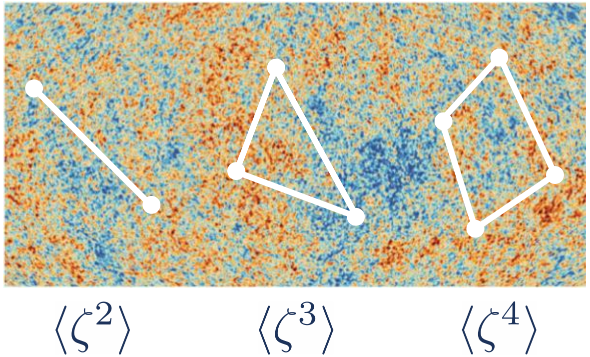
- At CMB scales the typical fluctuations are

$$P_\zeta \equiv H^2/(2\epsilon M_{\rm Pl}^2) \sim 10^{-10}, \ \ \zeta \sim 10^{-5}$$

↑ Power spectrum

Perturbation theory

Statistics of ζ is almost perfectly **Gaussian**, with corrections characterized by $\langle\zeta^3\rangle$, $\langle\zeta^4\rangle$



- Corrections to Gaussianity for $\zeta \sim P_\zeta^{1/2}$ (typical fluctuations)

$$\frac{\langle\zeta^3\rangle}{P_\zeta^2}\zeta \sim f_{\rm NL}\zeta \sim f_{\rm NL}P_\zeta^{1/2} \ll 1$$

$$\frac{\langle\zeta^4\rangle}{P_\zeta^3}\zeta^2 \sim g_{\rm NL}\zeta^2 \sim g_{\rm NL}P_\zeta \ll 1$$

(Planck and LLS bounds) $\lesssim 10^{-3}$

- Inflationary correlators are thus reliably computed in perturbation theory: (in-in formalism)

$$\langle\hat{Q}(\eta)\rangle = \langle 0|\bar{\mathrm{T}}e^{i\int_{-\infty(1-i\epsilon)}^{\eta}\hat{\mathrm{H}}^{\mathrm{I}}_{\mathrm{int}}(\eta')\mathrm{d}\eta'}\hat{Q}^{\mathrm{I}}(\eta)\mathrm{T}e^{-i\int_{-\infty(1-i\epsilon)}^{\eta}\hat{\mathrm{H}}^{\mathrm{I}}_{\mathrm{int}}(\eta'')\mathrm{d}\eta''}|0\rangle$$

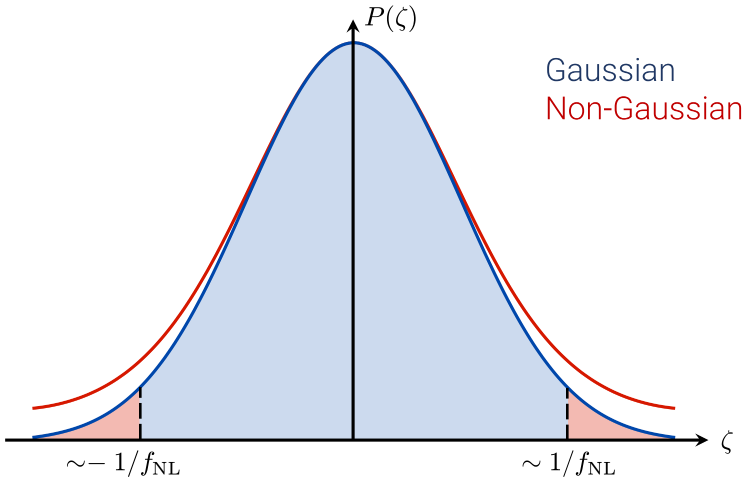
Why going beyond PT

- PT computes corrections close to the peak of the probability distribution $P(\zeta)$

It breaks down on the tails $f_{\rm NL}\zeta \sim 1$

$$P(\zeta) \sim \exp\left[-\frac{\zeta^2}{2P_\zeta} + \frac{\langle\zeta^3\rangle}{P_\zeta^3}\zeta^3 + \frac{\langle\zeta^4\rangle}{P_\zeta^4}\zeta^4 + \dots\right]$$
$$\sim \exp\left[-\frac{\zeta^2}{2P_\zeta}\left(1 + \frac{\langle\zeta^3\rangle}{P_\zeta^2}\zeta + \frac{\langle\zeta^4\rangle}{P_\zeta^3}\zeta^2 + \dots\right)\right]$$

Corrections depend on the size of ζ



- This regime can be relevant for the abundance of rare objects: **Primordial Black Holes**, CMB spots ecc..

BH mass fraction at formation

$$\beta(M) = \int_{\zeta_c}^{\infty} P(\zeta)\mathrm{d}\zeta\ , \quad \zeta_c \sim 1$$

Relevant for models of inflation with large NGs, such as k-inflation. In slow-roll instead $f_{\rm NL} \sim \mathcal{O}(\epsilon, \eta) \ll 1$

How to go beyond PT

The tail of the distribution is amenable to a semiclassical calculation

- For $\hbar \to 0$ fluctuations go to zero: intuitively this limit describes **unlikely events**

$$\Psi[\zeta_0(\vec{x})] = \int_{\rm BD}^{\zeta_0(\vec{x})} \mathcal{D}\zeta e^{iS[\zeta]/\hbar} \,, \quad S[\zeta]/\hbar \gg 1$$

Wavefunction of the Universe $|\Psi[\zeta]|^2 = P(\zeta)$

- This is the **semiclassical regime**

$$\Psi[\zeta_0(\vec{x})] \sim e^{iS[\zeta_{\rm cl}]/\hbar}$$

- We can see this explicitly in an example in QM: tails of the wavefunction cannot be described in PT

Semiclassical wavefunction in QM

- Consider a particle with position $x(t)$ in a potential well with potential $V(x)$

  We are interested in the ground state wavefunction $\Psi_0(x_f)$

- After rotating to **Euclidean time** $t \to -i\tau$, the ground state can be written as a path integral ($T \equiv \tau_f - \tau_i$)

$$\Psi_0\left(x_f\right)\Psi_0^*\left(x_i\right)e^{-E_0 T} = \lim_{T\to\infty}\int_{x(\tau_i)=x_i}^{x(\tau_f)=x_f} \mathcal{D}x(\tau)e^{-S_{\mathrm{E}}[x(\tau)]/\hbar}$$

↑ Selects the trajectory with $E=0$

- For large x_f we are on a tail of the wavefunction. The action is large: **semiclassical limit holds**

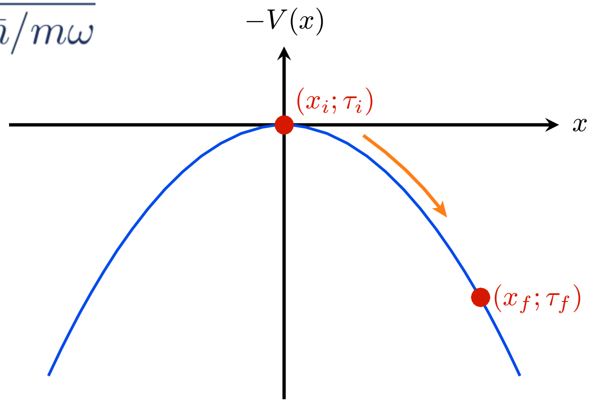
$$\Psi_0\left(x_f\right)\Psi_0^*\left(x_i\right)e^{-E_0 T} \sim e^{-S_{\mathrm{E}}[x_{\mathrm{cl}}(\tau)]/\hbar} \qquad S_{\mathrm{E}} = \int_{\tau_i}^{\tau_f}\left[\frac{1}{2}m\dot{x}^2 + V(x)\right]\mathrm{d}\tau$$

- Wavefunction can be obtained from a “classical” trajectory connecting the initial and final point in an **inverted potential**

Wavefunction for an anharmonic oscillator

Small parameter

- Example: $V(x) = \hbar\omega \left[\frac{1}{2}\left(\frac{x}{d}\right)^2 + \lambda \left(\frac{x}{d}\right)^4 \right] \,, \qquad d \equiv \sqrt{\hbar/m\omega}$



- The semiclassical parameter $\bar{x}^2 \equiv 2\lambda x_f^2/d^2$ can become large, so PT breaks down when $\bar{x}^2 \sim \mathcal{O}(1)$

- Because of **energy conservation** $(E = 0)$ the action is easy to find

$$\frac{S_\mathrm{E}[x(\tau)]}{\hbar} = \frac{1}{\hbar}\int_{\tau_i}^{\tau_f} m\dot{x}^2 \,\mathrm{d}\tau = \frac{1}{6\lambda}\left[\left(1+\bar{x}^2\right)^{3/2} - 1\right]$$

Non-perturbative result in λ

- The wavefunction has the form

One-loop correction

$$\Psi_0(\bar{x}) = \mathcal{N} \exp\left\{ -\frac{1}{6\lambda}\left[\left(1+\bar{x}^2\right)^{3/2} - 1\right] + f(\bar{x}) + \lambda g(\bar{x}) + \ldots \right\}$$

Two loops

Wavefunction for Inflation

- For Inflation, we consider a model where nonlinearities are dominated by a single term

$$S = \int \mathrm{d}^3x \mathrm{d}\eta \left\{ \frac{1}{2\eta^2 P_\zeta} \left[\zeta'^2 - (\partial_i \zeta)^2 \right] + \frac{\lambda \zeta'^4}{4! P_\zeta^2} \right\}$$

[Senatore, Zaldarriaga, '11]

- Standard perturbation theory: expansion in $\lambda \ll 1$

- The (classical) nonlinear parameter is $\tilde{\zeta}_0 \equiv \lambda^{1/2} \zeta_0 / P_\zeta^{1/2}$ Value of ζ at late times

  (analogous to $\bar{x} \equiv 2\lambda x_f^2 / d^2$ in QM)

- **Semiclassical expansion**: expansion in λ with $\tilde{\zeta}_0$ arbitrary.

  The on-shell action thus scales as

$$S = \frac{1}{\lambda} F(\tilde{\zeta}_0)$$

Wavefunction for Inflation

- The EoM in **Euclidean** ($\eta \to -i\tau$) is

$$\zeta'' - \frac{2}{\tau}\zeta' + \nabla^2\zeta + \frac{\lambda}{2P_\zeta}\tau^2\zeta'^2\zeta'' = 0$$

- We solve the EoM numerically for different BCs

$$\zeta(\tau_{\rm i}, \vec{x}) = 0, \quad \zeta(\tau_{\rm f}, \vec{x}) = \zeta_0(\vec{x})$$

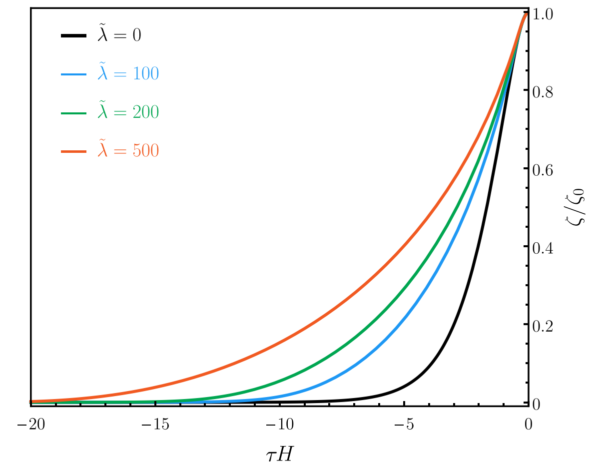
Large value

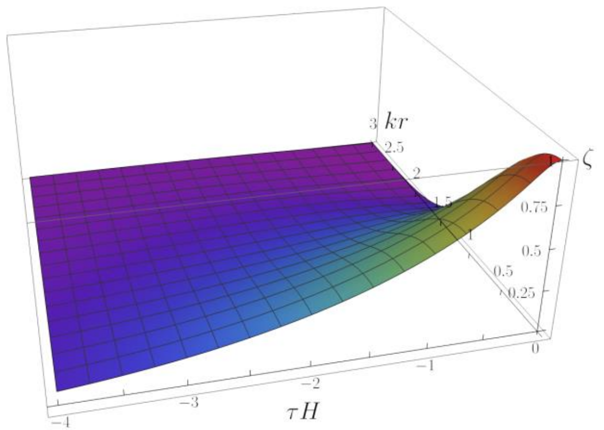
- We also need to fix the late-time configuration as a function of $\vec{x}$

- We choose a gaussian profile at late times

$$\zeta_0(\vec{x}) = \zeta_0 e^{-k^2 r^2}$$

$$\tilde{\lambda} = \lambda\zeta_0^2/P_\zeta$$





Wavefunction for Inflation

- After obtaining the solution, we can evaluate the Euclidean action
- The **free action** contains divergences at late times that we need to subtract:

$$\zeta_{\rm cl}(\vec{k},\tau) = \zeta_0(\vec{k})\frac{(1-k\tau)e^{k\tau}}{(1-k\tau_{\rm f})\,e^{k\tau_{\rm f}}}$$

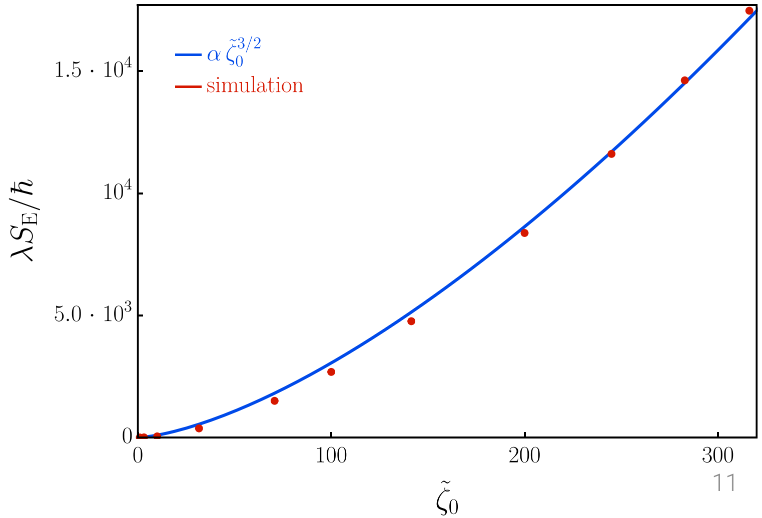
Divergent for $\tau_{\rm f} \to 0$ [Maldacena, '03]
Corresponds to a phase in Lorentzian
(irrelevant for the probability distribution)

$$S_{\rm E} = -\frac{1}{2P_\zeta}\int \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{\tau_{\rm f}^2}\zeta_{\rm cl}(-\vec{k},\tau)\partial_\tau\zeta_{\rm cl}(\vec{k},\tau)\Big|_{\tau=\tau_{\rm f}} \simeq \int \frac{{\rm d}^3k}{(2\pi)^3}\frac{1}{2P_\zeta}\left(\frac{k^2}{\tau_{\rm f}} + k^3 + \ldots\right)\zeta_0(-\vec{k})\zeta_0(\vec{k})$$

- In the nonlinear case, after subtracting the divergent part, we can numerically evaluate the action and get

$$S_{\rm E} \sim \frac{1}{\lambda}\tilde{\zeta}_0^{3/2}$$

$$\Psi\left[\zeta_0\right] \sim \exp\left[-\frac{1}{\lambda}\tilde{\zeta}_0^{3/2}\right]$$



Conclusions and future directions

Conclusions:

- We studied the tails of the probability distribution for ζ at late times
- In this regime usual PT breaks down. However, a semiclassical approach is possible
- We studied numerically this problem in a simple model by first rotating to Euclidean time

Future directions:

- This method can be applied to different models with large NGs (k-inflation, DBI, ecc..)
- More systematic study of PBH formation in these models

Conclusions and future directions

Conclusions:

- We studied the tails of the probability distribution for ζ at late times
- In this regime usual PT breaks down. However, a semiclassical approach is possible
- We studied numerically this problem in a simple model by first rotating to Euclidean time

Future directions:

- This method can be applied to different models with large NGs (k-inflation, DBI, ecc..)
- More systematic study of PBH formation in these models

Thank you for listening