Symmetries of Black Hole Perturbation Theory

Adam Solomon
Carnegie Mellon University

w/ Lam Hui, Austin Joyce, Riccardo Penco, & Luca Santoni
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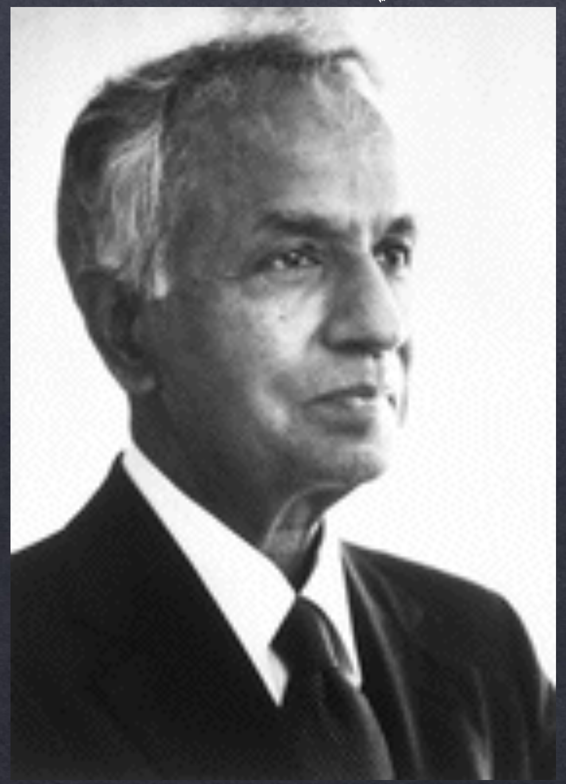
"The black holes of nature are the most perfect macroscopic objects there are in the universe: the only elements in their construction are our concepts of space and time."

-Chandrasekhar



Motivation

- Black holes are simple but mysterious
- "A new era of gravitational wave astronomy"™
- Binary BH mergers probe gravity in a wide variety of regimes
- This talk: linear perturbations; Schwarzschild for simplicity
  - Many/(all?) results apply to Kerr

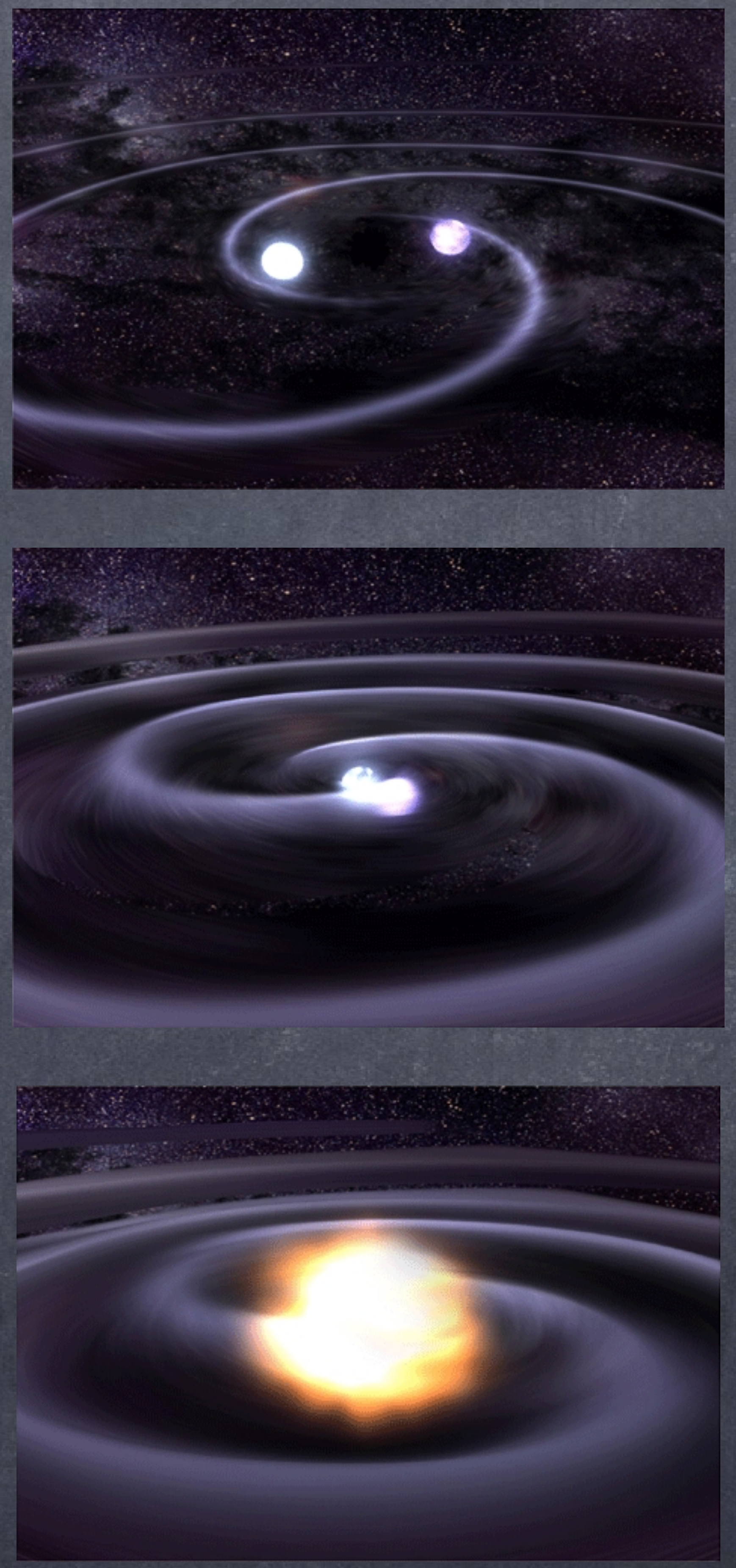


Image: NASA

Inspiral
Post-Newtonian; EFT; effective one-body; perturbation theory

Merger
Numerical relativity

Ringdown
Perturbation theory

Black Hole Spectroscopy

- Ringdown dominated by quasinormal modes
- QNMs: decaying waves with BCs:
  - Ingoing at horizon
  - Outgoing at infinity
- Discrete spectra: allows spectroscopy à la atoms
- Applications: no-hair (Isi+ 1905.00869), tests of GR, BH mimickers, ...

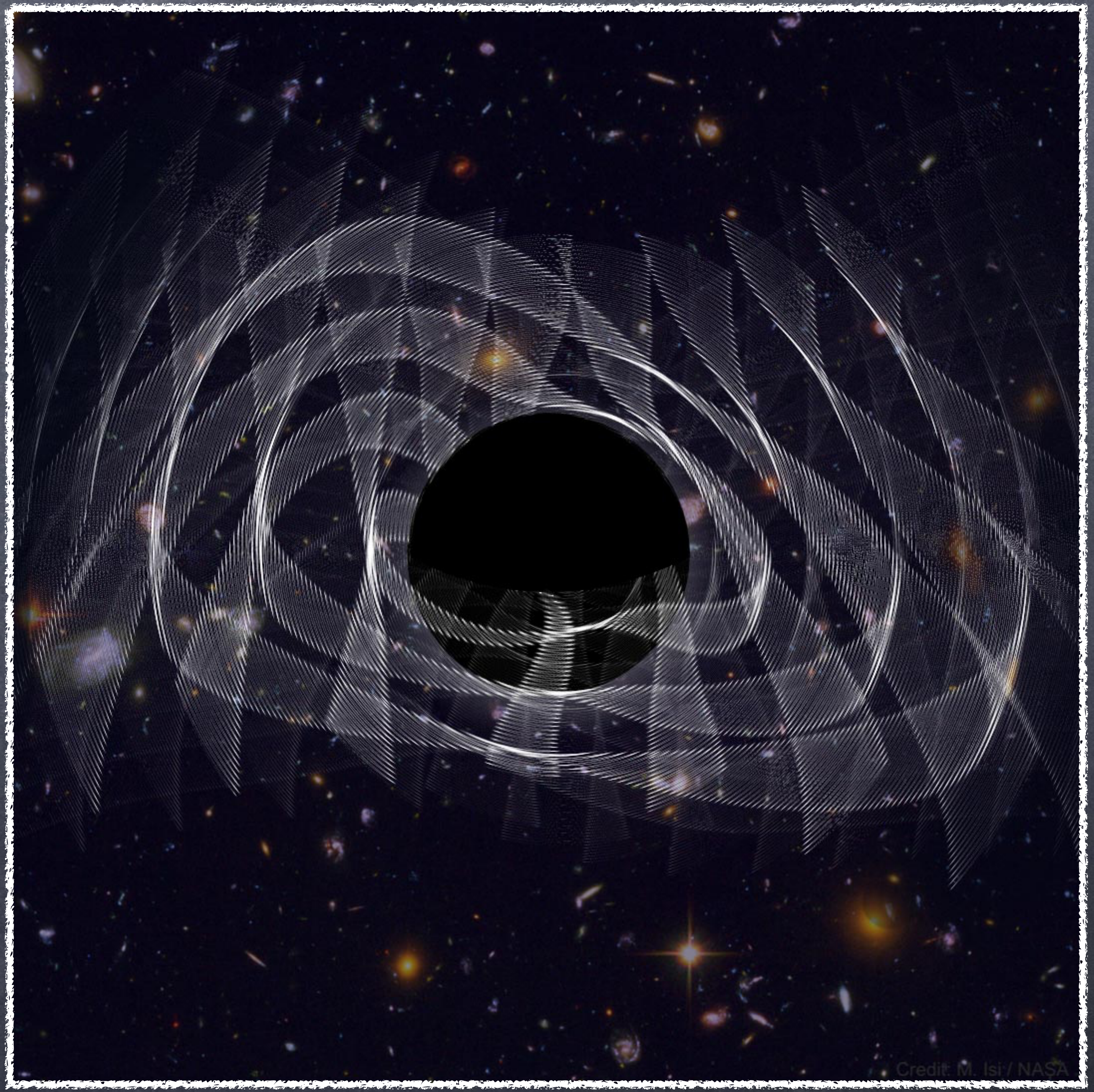


Image: M. Isi

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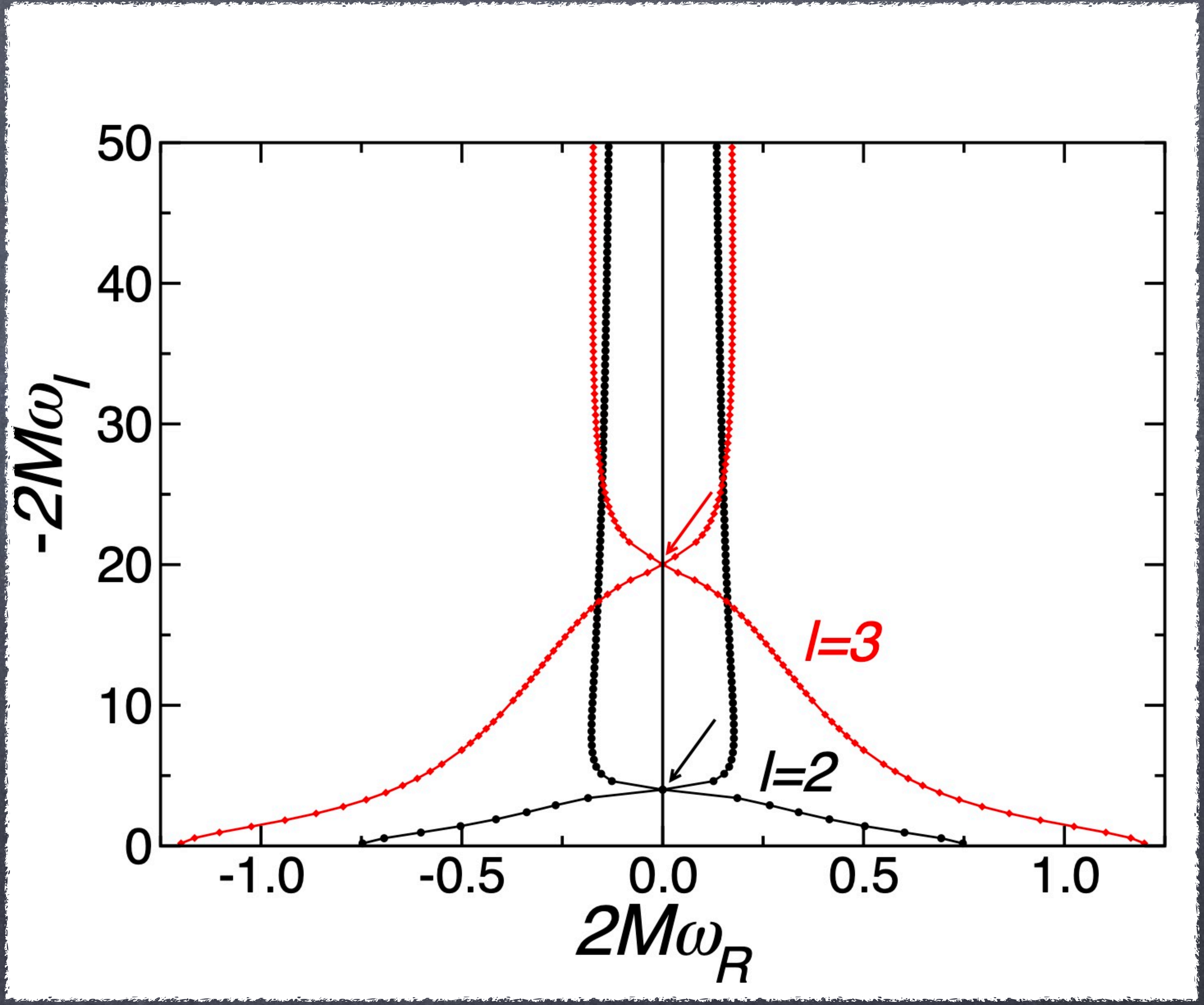


Figure: Berti, Cardoso, & Starinets (0905.2975, CQG review)

Tidal Responses & Love Numbers

- During inspiral, companion object sources a tidal field
- Gravitational response to tidal deformation encoded in Love numbers
  - Measures internal structure
  - Observable at 5PN
- Wilson coefficients of point-particle EFT encoding finite size/structure:

$$S_{\text{pp}} = \int \mathrm{d}\tau \left(-m + \frac{1}{2}\lambda_E E_{ij}^2 + \frac{1}{2}\lambda_B B_{ij}^2 + \cdots \right)$$

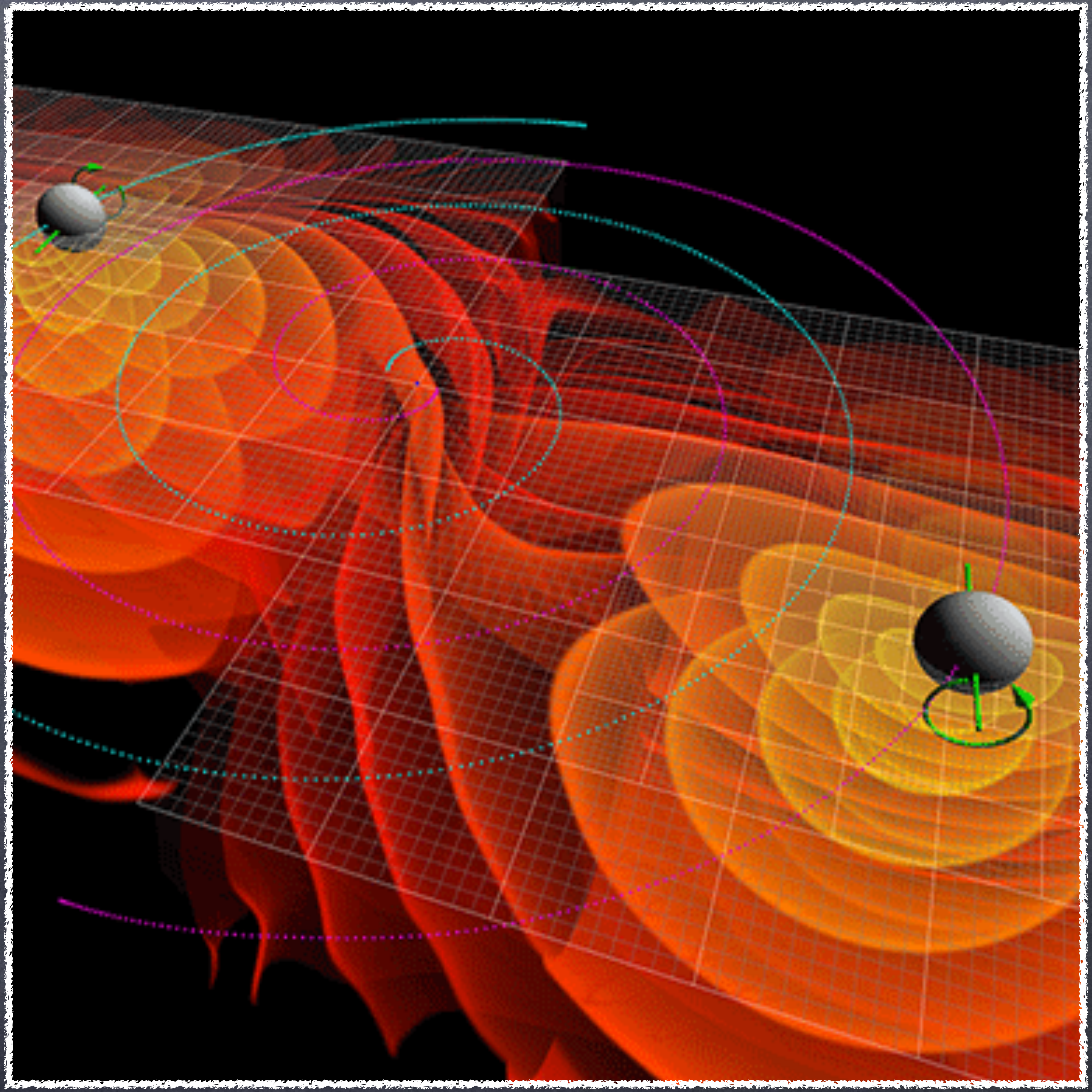


Image: C. Henze/NASA Ames Research Center

Hints of Symmetries
(in D=4 GR)

Symmetry of Love? Black hole Love numbers = 0

"Fine tuning": vanishing EFT coefficients (Porto, 1606.08895)

Symmetry of spectra? Both GW polarizations have the same QNM spectrum

Caused by a duality of the Einstein equations

A Black Hole Perturbation Theory Primer

(for cosmologists)

Expand metric around background

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_{\rm Pl}} h_{\mu\nu}$$

Split by behavior under parity

(viz scalar/vector/tensor decomposition)

$$h_{\mu\nu} = h^{\rm even}_{\mu\nu} + h^{\rm odd}_{\mu\nu}$$

Decompose into (m=0) spherical harmonics

(viz Fourier expansion)

$$h_{\mu\nu}(t, r, \theta, \phi) = \sum_\ell h^\ell_{\mu\nu}(t, r)\Theta(\theta)$$

$Y_{\ell 0}, \sin\theta\partial_\theta Y_{\ell 0}$

Graviton has 2 dof: 1 in even sector and 1 in odd

At infinity, these correspond to +/x polarizations

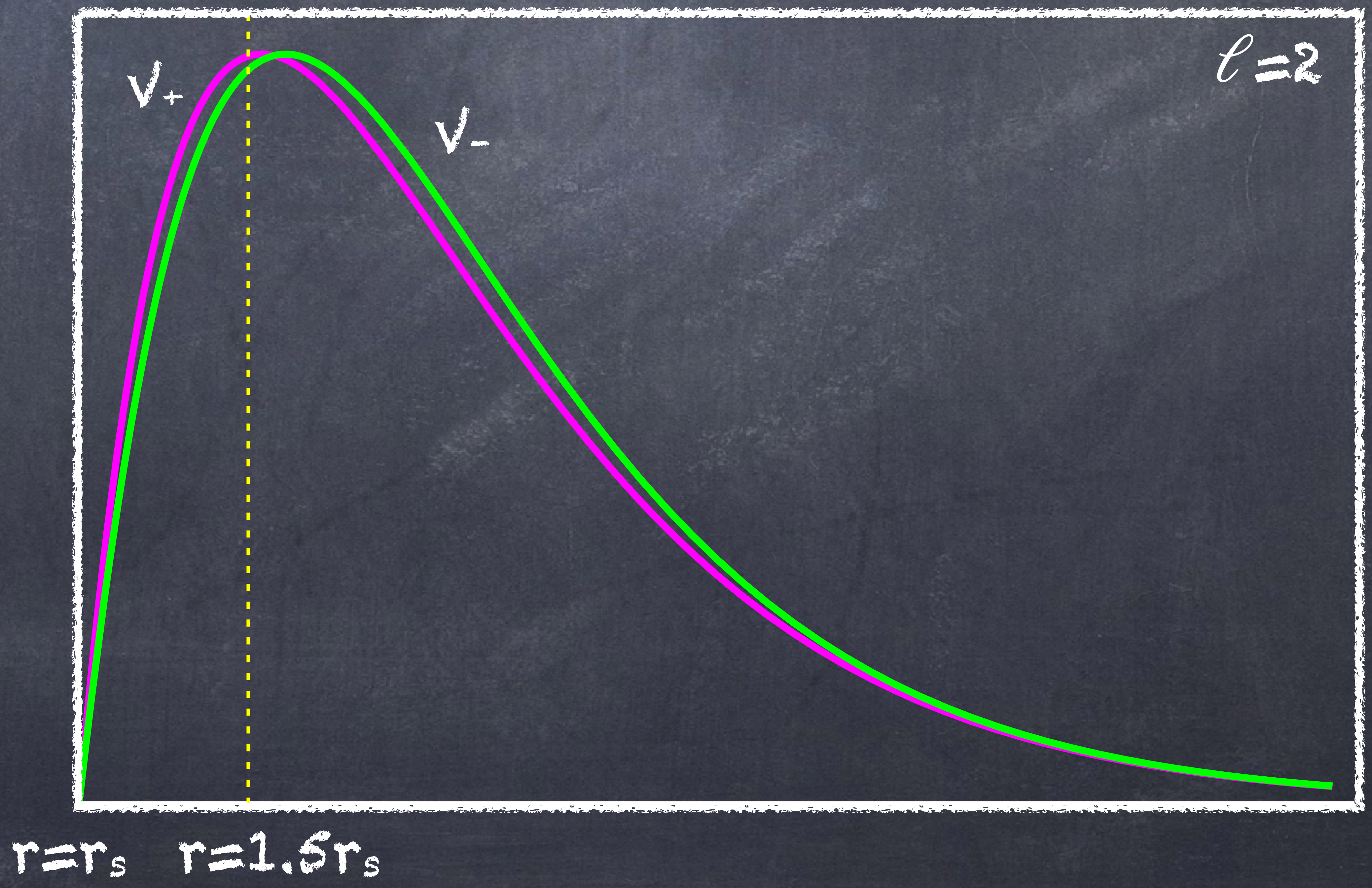
Regge-Wheeler and Zerilli

- Encode dynamical d.o.f. in **master variables**:
  - Even: Zerilli Ψ_+
  - Odd: Regge-Wheeler Ψ_-
- These obey simple **Schrödinger-like** equations
- Tortoise coordinate:

$$\mathrm{d}r \equiv \left(1 - \frac{r_s}{r}\right) \mathrm{d}r_\star$$

Master equation:

$$\frac{\partial^2 \Psi_\pm(t,r)}{\partial t^2} - \frac{\partial^2 \Psi_\pm(t,r)}{\partial r_\star^2} + V_\pm(r)\Psi_\pm(t,r) = 0$$



Chandrasekhar's Duality

- Secret link between Regge-Wheeler and Zerilli potentials:

$$V_{\pm}(r) = W^2(r) \mp \frac{\mathrm{d}W(r)}{\mathrm{d}r_\star} + \beta$$

with W(r) the superpotential and β a constant

- This directly implies isospectrality

Chandrasekhar (1980s)

$$V_+ = \frac{1-\frac{r_s}{r}}{r^3}\frac{9r_s^3 + 12\lambda^2 r_s r^2 + 8\lambda^2(1+\lambda)r^3 + 18\lambda r_s^2 r}{(2\lambda r + 3r_s)^2}$$

$$2\lambda \equiv (\ell - 1)(\ell + 2)$$

$$V_- = \left(1 - \frac{r_s}{r}\right)\left(\frac{\ell(\ell+1)}{r^2} - \frac{3r_s}{r^3}\right)$$

Our question: where does this property in GR come from?

Linearized Einstein-Hilbert

To study symmetries, we want to work at the level of the action:

$$S = \frac{M_{\mathrm{Pl}}^2}{2} \int \mathrm{d}^4x \sqrt{-g} R \bigg|_{g_{\mu\nu} = \bar{g}_{\mu\nu} + \frac{1}{M_{\mathrm{Pl}}} h_{\mu\nu}}$$

Why? Allows us to calculate e.g. Noether currents

Procedure

1. Fix gauge
2. Integrate out non-dynamical (auxiliary) dofs
3. Canonically normalize: rescale field and coordinates

parity and spherical harmonics decoupling:

$$S = \sum_{\pm} \sum_{\ell} \int \mathrm{d}t \mathrm{d}r \mathscr{L}_{\pm}^{\ell}(t, r)$$

Linearized Einstein-Hilbert

Canonical normalization

- Coordinate choice: $r \rightarrow r_\star$
- Field redefinition: $h^{\pm}_{\mu\nu} \rightarrow \Psi_\pm$

End result

$$S = \frac{1}{2}\sum_{\ell=2}^{\infty}\int \mathrm{d}t\mathrm{d}r_\star \sum_{\pm}\left[\left(\frac{\partial\Psi_\pm}{\partial t}\right)^2 - \left(\frac{\partial\Psi_\pm}{\partial r_\star}\right)^2 - V_\pm(r)\Psi_\pm^2\right]$$

Chandrasekhar Duality in Action

Replacing the potentials with the superpotential:

$$S = \frac{1}{2}\sum_{\ell=2}^{\infty}\int \mathrm{d}t\mathrm{d}r_\star \sum_{\pm}\left[\left(\frac{\partial\Psi_\pm}{\partial t}\right)^2 - \left(\frac{\partial\Psi_\pm}{\partial r_\star}\right)^2 - \left(W^2 \mp \frac{\mathrm{d}W}{\mathrm{d}r_\star} + \beta\right)\Psi_\pm^2\right]$$

the action is invariant under

$$\delta\Psi_\pm = \left(\frac{\partial}{\partial r_\star} \mp W(r)\right)\Psi_\mp$$

We see that Chandrasekhar duality is a symmetry of Einstein-Hilbert (i.e., off-shell)

(NB: can also write symmetry at the level of metric perturbations)

Application: Tidal Response

- Noether current for static solutions:

$$J^{r_\star} = \partial_{r_\star}\Psi_+\partial_{r_\star}\Psi_- + W(\Psi_+\partial_{r_\star}\Psi_- - \Psi_-\partial_{r_\star}\Psi_+) - (W^2+\beta)\Psi_+\Psi_- = \text{const}\,.$$

- Regularity at the horizon: $J^{r_\star} = 0$
- At infinity: $\Psi_\pm \propto r^{\ell+1} + \lambda_\pm r^{-\ell}$: $J^{r_\star} \propto (\lambda_+ - \lambda_-)$
- Duality implies equal Love numbers
- Not quite vanishing, but helpful: odd sector much simpler than even

Minkowski Limit

Electric-magnetic duality

Duality in the flat limit:

SO(2)

$$\delta\Psi_+ = -\Psi_-$$
$$\delta\Psi_- = \Psi_+$$

implies, on-shell,

Gravitational electric-magnetic duality:

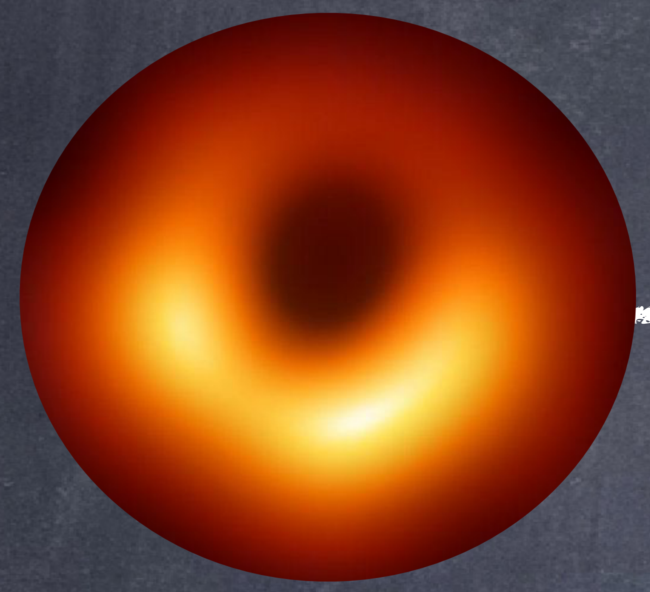
$$\delta R_{\mu\nu\alpha\beta} = \tilde{R}_{\mu\nu\alpha\beta}$$
$$\delta\tilde{R}_{\mu\nu\alpha\beta} = -R_{\mu\nu\alpha\beta}$$

$$\tilde{R}_{\mu\nu\alpha\beta} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma}R^{\rho\sigma}{}_{\alpha\beta}$$

EM duality previously known symmetry of Einstein-Hilbert around Minkowski, (A)dS

Vanishing Love, Mathematically

static Regge-Wheeler: $\partial_{r_\star}^2 \Psi - V(r)\Psi = 0$



Horizon — **Infinity**

regular	$\psi \sim \text{const.}$	$\longleftrightarrow$	growing	$\psi \sim r^{\ell}$	(tidal field)
~~blowing~~ up	$\psi \sim \ln(1 - r_s/r)$		decaying	$\psi \sim \dfrac{1}{r^{\ell+1}}$	(static response)

Love number: coefficient of decaying term

Math: why do these diff eqs have this property?

Physics: why does GR give us diff eqs like this?

See also Charalambous, Dubovsky, and Ivanov (2103.01234)

Spin-0 Warm-Up

Klein-Gordon on Schwarzschild: $\nabla^2\phi = 0 \longrightarrow \sum_\ell H_\ell \phi_\ell = 0$

$$H_\ell = -\Delta\left[\partial_r(\Delta\partial_r) - \ell(\ell+1)\right]$$

$\Delta(r) \equiv r(r - r_s)$

Admits raising and lowering operators:

$$D_\ell^+ \equiv -\Delta\partial_r + \frac{\ell+1}{2}(r_s - 2r)$$

$$D_\ell^- \equiv \Delta\partial_r + \frac{\ell}{2}(r_s - 2r)$$

in the sense that

$$H_{\ell+1}(D_\ell^+\phi_\ell) = 0$$

$$H_{\ell-1}(D_\ell^-\phi_\ell) = 0$$

Note: This example contains the salient features of spin-s on Kerr

Turning the Ladder Sideways

- The raising/lowering symmetry $D_\ell^\pm$ is unusual: it relates solutions at different levels ℓ
- Want: a symmetry for each level individually
- Strategy: lower to $\ell = s$ and use translation symmetry
- Horizontal symmetry: $\delta\phi_\ell = Q_\ell\phi_\ell$
- Conserved charge: $P_\ell \equiv \Delta\partial_r(D_1^- D_2^- \cdots D_\ell^- \phi_\ell)$

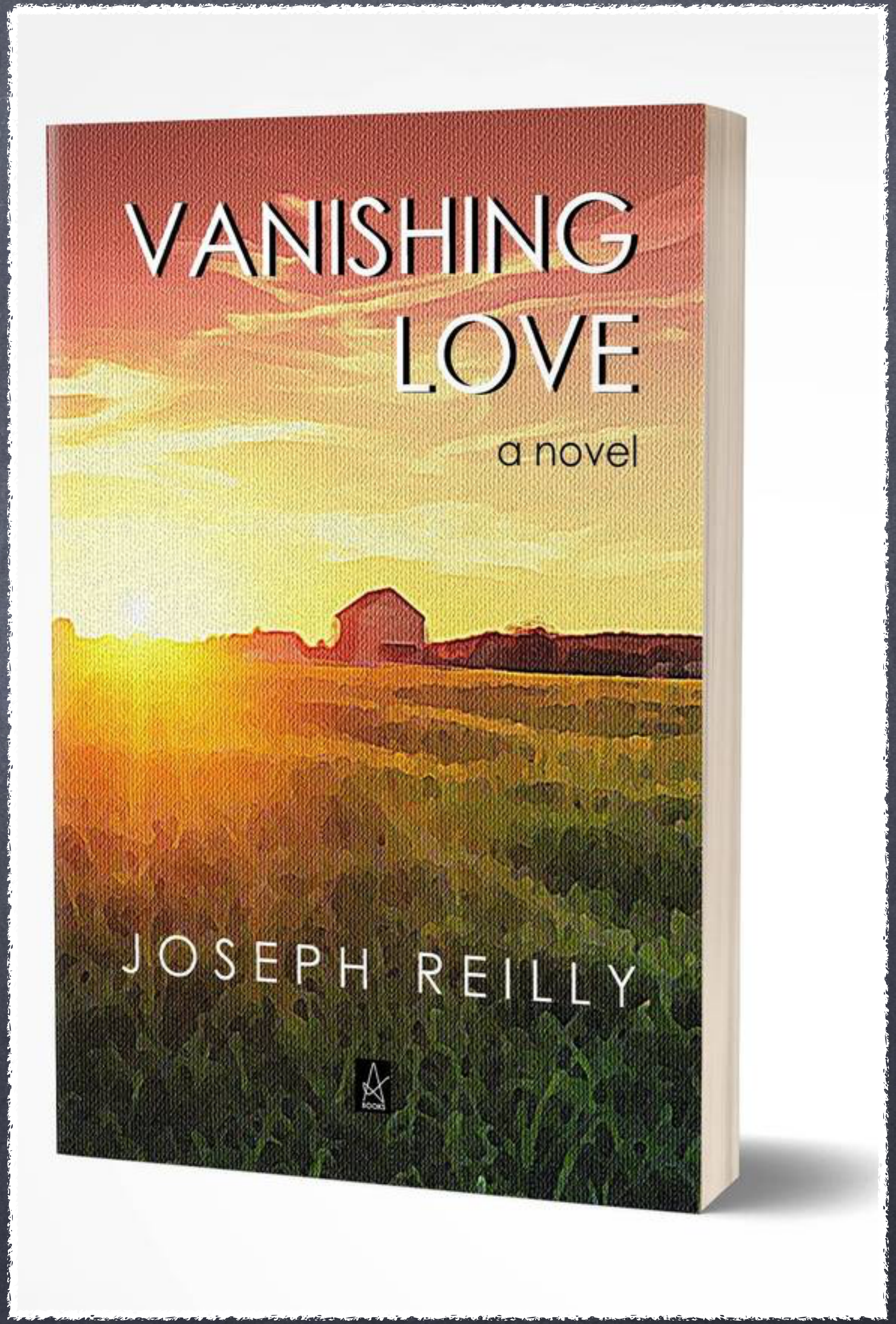
$$P_0 = \Delta\partial_r\phi_0$$

$$Q_0 \equiv \Delta\partial_r$$
$$Q_1 \equiv D_0^+ Q_0 D_1^-$$
$$\vdots$$
$$Q_\ell \equiv D_{\ell-1}^+ Q_{\ell-1} D_\ell^-$$

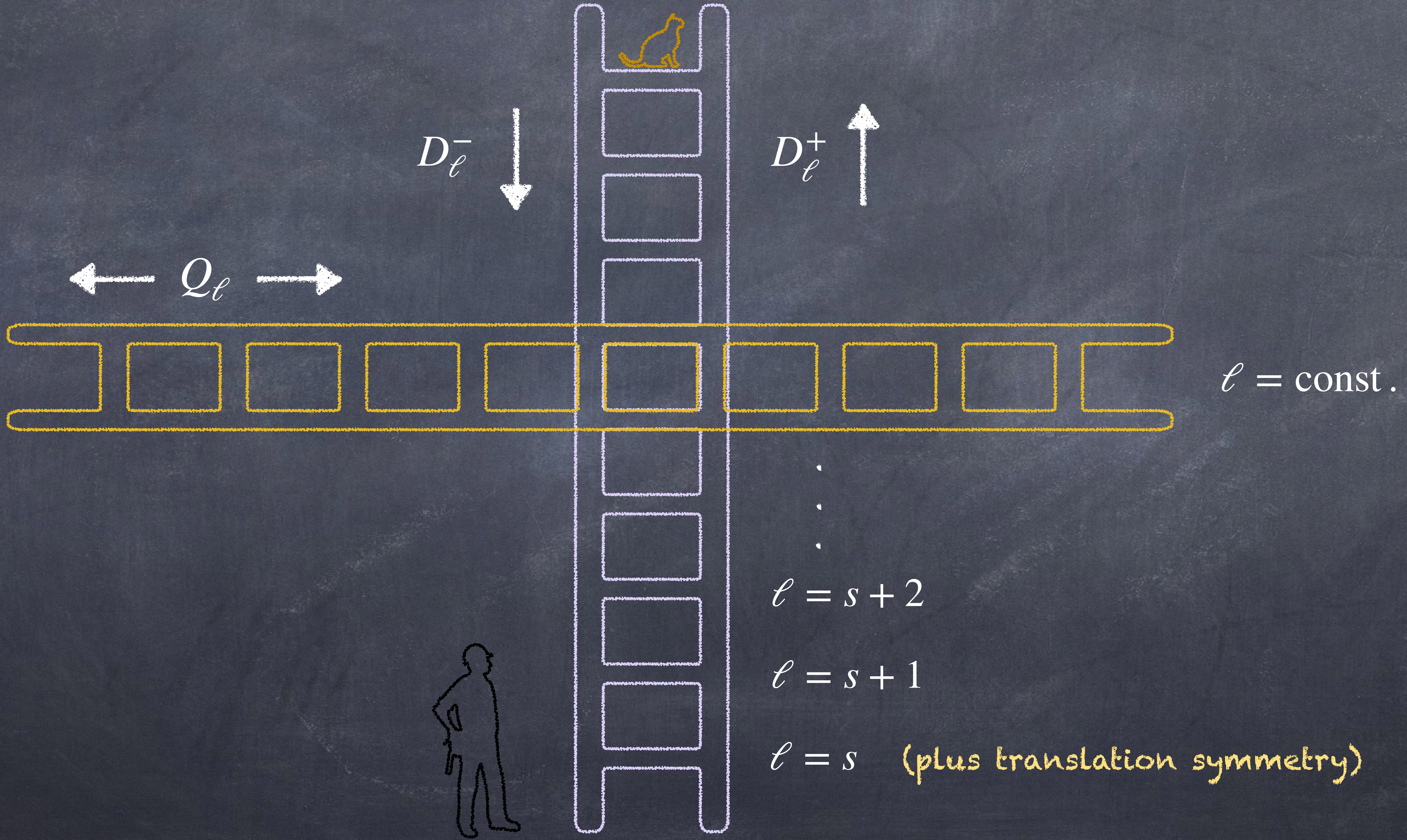
See also Compton and Morrison (2003.08023)

Vanishing Love

- Charge conservation implies Love = 0
  - $P_\ell = 0$ for growing (infinity) and constant (horizon) modes
  - $P_\ell \neq 0$ for decaying (infinity) and divergent (horizon) modes
- Spontaneous symmetry breaking: $Q_\ell \phi_\ell^{(\mathrm{g})} = 0, \quad Q_\ell \phi_\ell^{(\mathrm{d})} \neq 0$
- Conclusion: a decaying term diverges at the horizon
  - This also implies no (static) hair



Ladders Up and Down



Geometric Interpretation

From Schwarzschild to AdS

Static scalar: $S = \frac{1}{2}\int \mathrm{d}^3x\sqrt{g}\,\phi\,\Box\,\phi$

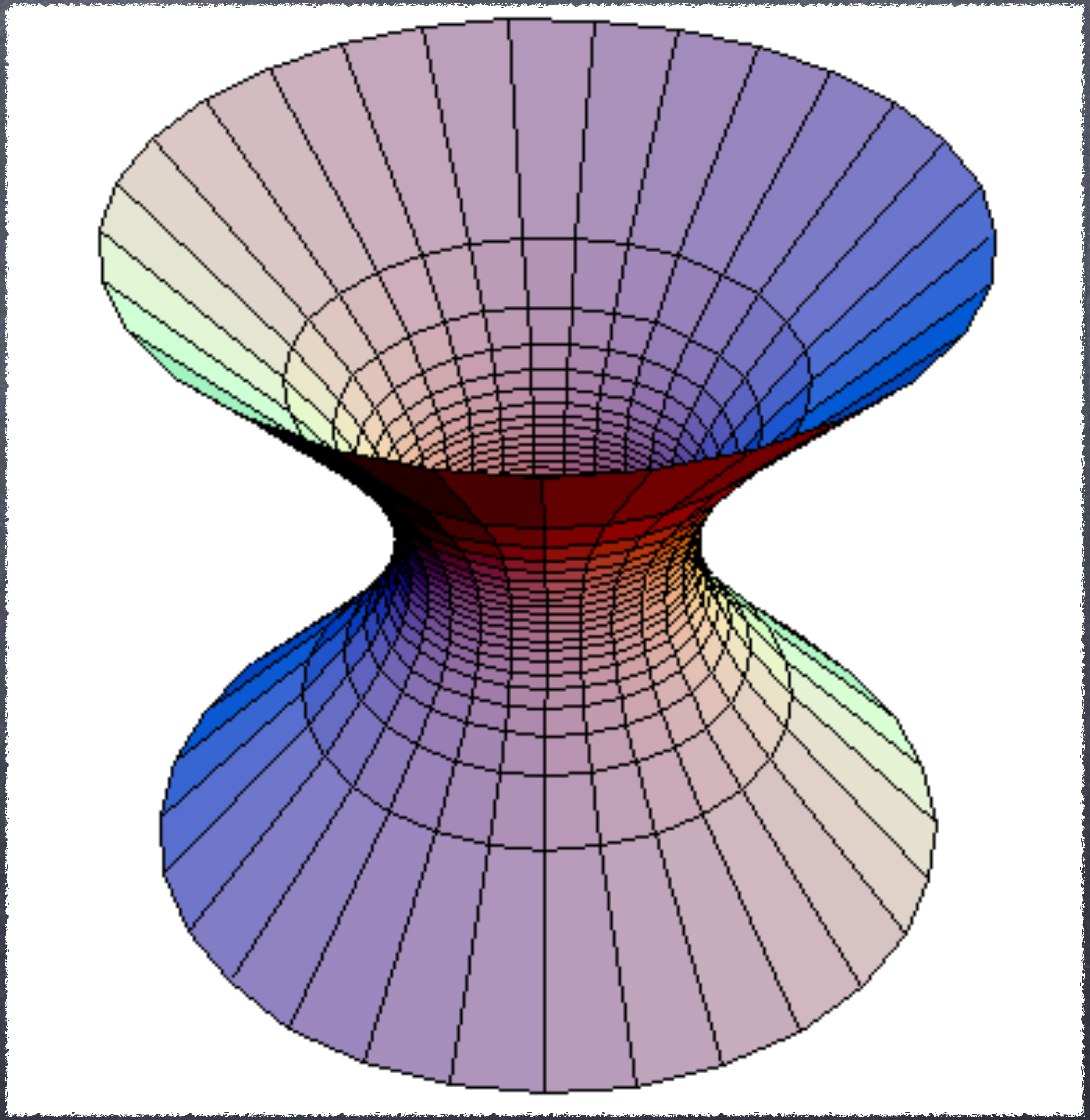
Conformal transformation:

$\tilde{g}_{ij} = L^4\Delta^{-2}g_{ij}, \quad \tilde{\phi} = L^{-1}\sqrt{\Delta}\,\phi$ L: arbitrary scale

so that

$$S = \frac{1}{2}\int \mathrm{d}^3x\sqrt{\tilde{g}}\left(\tilde{\phi}\,\tilde{\Box}\,\tilde{\phi} + \frac{r_s^2}{4L^4}\tilde{\phi}^2\right)$$

The metric $\tilde{g}_{ij}$ is nothing other than AdS_3



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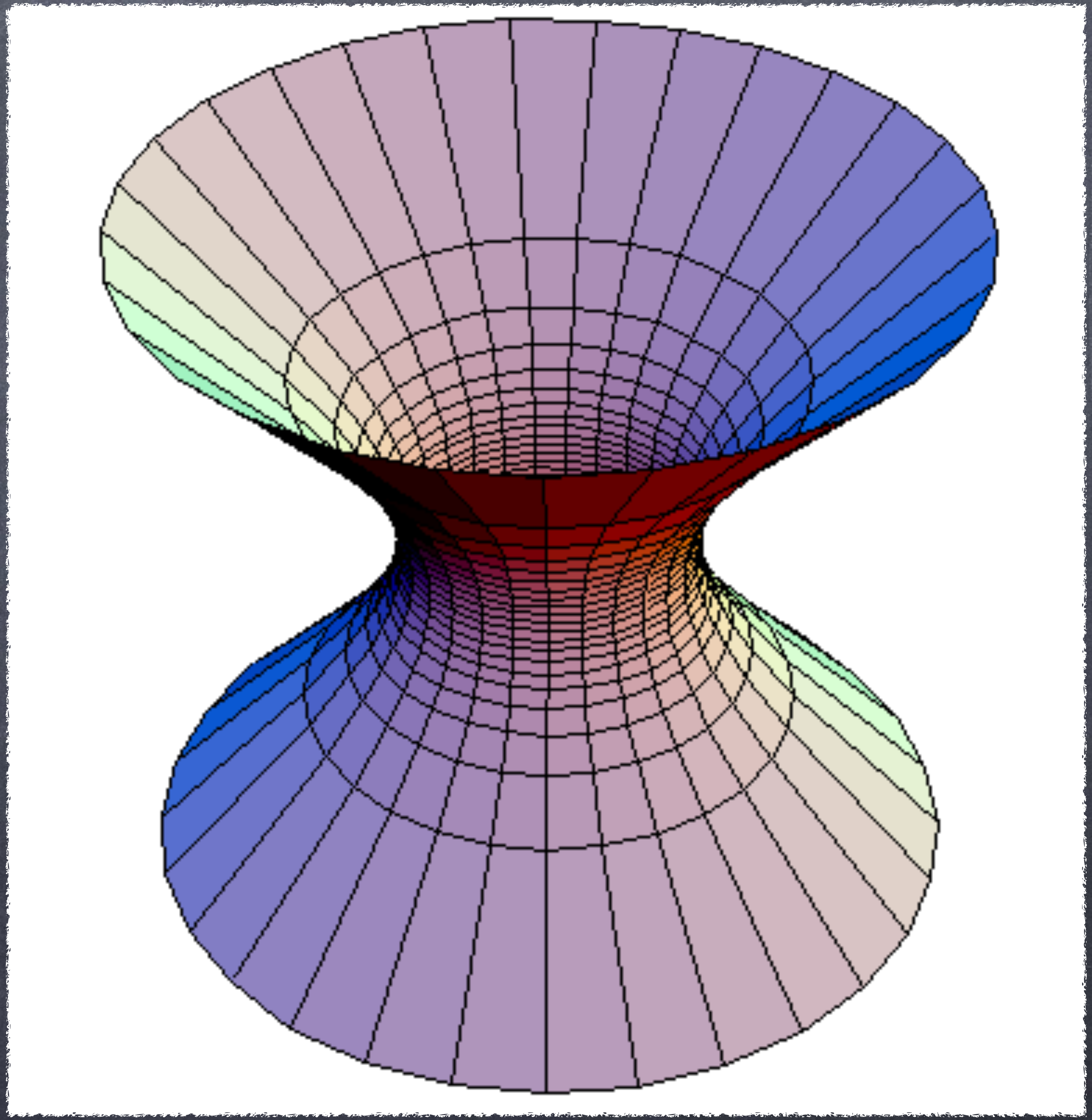
$$S = \frac{1}{2}\int \mathrm{d}^3x\sqrt{\tilde{g}}\left(\tilde{\phi}\,\Box\,\tilde{\phi} + \frac{r_s^2}{4L^4}\tilde{\phi}^2\right)$$

Killing vectors of AdS_3 generate isometry:

$$\delta\phi = -2\Delta\cos\theta\partial_r\phi + (r_s - 2r)\partial_\theta(\sin\theta\phi)$$

Decompose in spherical harmonics:

$$\delta\phi_\ell \sim D^+_{\ell-1}\phi_{\ell-1} + D^-_{\ell+1}\phi_{\ell+1}$$



To Kerr: Spin Ladder

Teukolsky equation:

$$\partial_r\left(\Delta\partial_r\phi_\ell^{(s)}\right) + s(2r - r_s)\partial_r\phi_\ell^{(s)} + \left(\frac{a^2m^2 + is(2r - r_s)am}{\Delta} - (\ell - s)(\ell + s + 1)\right)\phi_\ell^{(s)} = 0$$

Admits ladders in ℓ and spin!

$$E^+ \equiv \partial_r\,, \quad E_s^- \equiv \Delta\partial_r - s(r_+ + r_- - 2r) - 2iam\frac{r_+ - r_-}{r_+ - r_-}$$

Relates solutions to Klein-Gordon, Maxwell, and Einstein

$\boxed{\text{vanishing scalar Love} \longrightarrow \text{vanishing gravitational Love}}$

NB: also have ladders for Regge-Wheeler and Teukolsky for more direct proof

IR symmetries

Point-particle EFT:
treat BH as a point, encode structure in higher-order operators

γ: worldline einbein
g: (monopole) scalar charge
λ_ℓ: Love numbers

$$S = -\frac{1}{2}\int \mathrm{d}^4x(\partial\phi)^2 + \int \mathrm{d}\tau\gamma\left[\frac{1}{2}\gamma^{-2}\dot{x}^\mu\dot{x}_\mu - \frac{\mu^2}{2} - g\phi + \sum_{\ell=0}^{\infty}\frac{\lambda_\ell}{2\ell!}\left(\partial_{(a_1}\cdots\partial_{a_\ell)_T}\phi\right)^2\right]$$

UV symmetry in flat-space limit: $\delta\phi = r^2\cos\theta\partial_r\phi + r\partial_\theta(\sin\theta\phi)$

$\boxed{\text{Punchline: only the bulk } (\partial\phi)^2 \text{ term is invariant}}$

Summary

- Vanishing Love numbers and isospectral QNMs both indicate hidden symmetries of GR (/massless fields on Schwarzschild)
- We find symmetries of Einstein-Hilbert underlying these:
  - QNMs: EM duality on Schwarzschild
  - Love = 0: ladder symmetries (shift sym + conformal Killing vector/ladder syms)