Quantum Cosmological Correlation Functions and Their Adiabatic Regularization

A Talk for Cosmology from Home Conference

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What/Why of Quantum Cosmological Perturbations

Our current understanding is that a description of the very early universe should involve a quantum theory of gravity, but for now we do not have one that is agreed upon.

The next easy thing we can do is to take our best theory of gravity i.e. Einstein's gravity and perturb it around a fixed background. Then we can quantize the perturbation fields. This will be in effect a quantum field theory (QFT) on a curved space-time.

This method is justified by observations: Expectation values of such quantum fields around a inflationary background can be shown to consistently describe the current observed variations in the CMB sky and also the large scale structure in the universe.

In the following I will describe how to calculate quantum expectation values in such a theory. As expected from a QFT, these values will have divergences. For regularization of these divergences I will employ adiabatic regularization scheme,suggested by Leonard Parker.

Originally this scheme was used for regularization of QFT in curved background, where qravity is unperturbed and purely classical. Here we will show what happens if we use it for a theory where

perturbations of the metric is quantized,

and an expectation value at point of time is calculated by an in-in formulation.

Outline

1 Motivation

2 Framework for Quantum Cosmology

3 Expectation Values

4 Adiabatic Regularization

5 Getting Counterterms from Adiabatic Regularization

Consider the Einstein Hilbert action with a (minimally coupled) scalar field

$$S[g,\phi] = \int d^4x\sqrt{-g}\,(R(g) + \mathcal{L}_M(g,\phi)) \tag{1}$$

.Perturbing the fields

$$S[\bar{g}+g,\bar{\phi}+\varphi] = \int d^4x\sqrt{-(\bar{g}+g)}\,\left(R(\bar{g}+g) + \mathcal{L}_M(\bar{g}+g,\bar{\phi}+\varphi)\right) \tag{2}$$

.Perturbing the fields

$$\begin{aligned} S[\bar{g}+g,\bar{\phi}+\varphi] = \int d^4x\sqrt{-\bar{g}}\big(&\mathcal{L}^{(0)}(\bar{g},\bar{\phi}) + \mathcal{L}^{(1)}(\bar{g},\bar{\phi},g,\varphi) \\ &+ \mathcal{L}_R^{(2)}(\bar{g},g) + \mathcal{L}_M^{(2)}(\bar{g},g,\bar{\phi},\varphi) \\ &+ \mathcal{L}_R^{(3)}(\bar{g},g) + \mathcal{L}_M^{(3)}(\bar{g},g,\bar{\phi},\varphi) + \cdots\big) \end{aligned} \tag{3}$$

.Perturbing the fields

$\bar{g}$, $\bar{\phi}$ bckgnd solutions

$$\begin{aligned} S[\bar{g}+g,\bar{\phi}+\varphi] = \int d^4x\sqrt{-\bar{g}}\big(&\mathcal{L}^{(0)}(\bar{g},\bar{\phi}) + \mathcal{L}^{(1)}(\bar{g},\bar{\phi},g,\varphi) \\ &+ \mathcal{L}_R^{(2)}(\bar{g},g) + \mathcal{L}_M^{(2)}(\bar{g},g,\bar{\phi},\varphi) \end{aligned}$$

Now we quantize the perturbations g, φ. These will be the seeds of the structure in the universe. To calculate a distribution on the sky like CMB, one needs to calculate N-point functions of fields at a given time. Such an expectation value, in the Schrödinger picture, Interaction picture is written as

$$\langle \varphi\varphi\cdots\varphi\rangle = \langle\psi_S|\,\varphi_S(t,x_1)\varphi_S(t,x_2)\cdots\varphi_S(t,x_N)\,|\psi_S\rangle \tag{6}$$

$$\langle \varphi\varphi\cdots\varphi\rangle = \langle\psi_I|\,\varphi_I\varphi_I\ldots\,|\psi_I\rangle \tag{7}$$

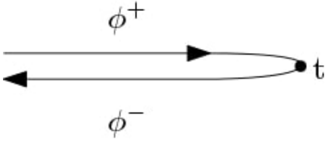
$$\begin{aligned}\langle \varphi\varphi\cdots\varphi\rangle &= \langle\psi_I|\,\varphi_I\varphi_I\ldots\varphi_I\,|\psi_I\rangle \\ &= \langle\psi_S(t_0)|\,\bar{T}\left(e^{i\int_{t_0}^{t} H_I'(t')dt'}\right)\varphi_I\varphi_I\ldots T\left(e^{-i\int_{t_0}^{t} H_I'(t')dt'}\right)|\psi_S(t_0)\rangle\end{aligned} \tag{8}$$

where φ_I are interaction picture fields, and H_I' is the interaction Hamiltonian in the interaction picture. (Interaction Hamiltonian is defined as the part of the Hamiltonian that is cubic and higher order in fields.)

If instead of canonical quantization one uses the path integral formulation then

$$\langle \varphi\varphi \cdots \varphi \rangle = \int D\phi(t) \int \prod_{t_0}^{t} D\phi^+ D\phi^- e^{iS[\phi^+]-iS[\phi^-]}$$
$$\phi(t,x_1)\phi(t,x_2)\cdots\phi(t,x_N)\Psi_0[\phi^+(t_0)]\Psi_0^*[\phi^-(t_0)] \qquad (9)$$

This has the picture



and hence the name in-in.

Having settled up this framework, let us use it to calculate a simple expectation value at the "tree" level: two point function. For this purpose let us consider a case in $1+3$ spacetime dimensions with two external fields where

$$\bar{g} = -dt^2 + a^2(t)dx^2\,, \quad \bar{\phi} \to \text{Slow-roll}\,, \quad \varphi \to \varphi, \chi\,, \tag{10}$$

and the matter Lagrangian is

$$\mathcal{L}_M^{(2)}(\bar{g}, \bar{\phi}, \varphi, \chi) = \frac{\dot{\bar{\phi}}^2}{2} - \frac{1}{2}\nabla_\mu\chi\nabla^\mu\chi - v(\bar{\phi}) - \frac{1}{2}m_\chi^2\chi^2 - \frac{1}{2}\kappa^2\bar{\phi}^2\chi^2 \tag{11}$$

Assume now then we would like to consider the two point function for χ. The full quantum expectation value is

$$\begin{aligned}\langle\chi(t,x_1)\chi(t,x_2)\rangle &= \langle\psi|\, \bar{T}\left(e^{i\int_{t_0}^{t} H_I'(t')dt'}\right)\chi_{I,1}\chi_{I,2}\, T\left(e^{-i\int_{t_0}^{t} H_I'(t')dt'}\right)|\psi\rangle \\ &= \langle\chi_{I,1}\chi_{I,2}\rangle + i\int_{t_0}^{t}\langle[\mathcal{H}_I'(t'),\chi_{I,1}\chi_{I,2}]\rangle \\ &+ i^2\int_{t_0}^{t}\int_{t_0}^{t''}\langle[\mathcal{H}_I(,t'),[\mathcal{H}_I(t''),\chi_{I,1}\chi_{I,2}]]\rangle \\ &+ \mathcal{O}(3)\,. \end{aligned} \tag{12}$$

First order term is then

$$\langle \chi_{I,1}\chi_{I,2}\rangle = \langle \psi_S(t_0)|\, \chi_I(t,x_1)\chi_I(t,x_2)\, |\psi_S(t_0)\rangle \ . \tag{13}$$

Remember χ_I is interaction picture operator, defined as

$$\chi_I(t,x) = {U^f}^{-1}\chi_S(x,t)U^f \tag{14}$$

where U^f is the evolution operator corresponding to the free-2nd order-part of the Hamiltonian. Solving this will be equivalent to solving

$$\ddot{\mu}_I(t,x) + \left(m_\chi^2 + \kappa^2\bar{\phi}^2 - \frac{9}{4}H^2 - \frac{3}{2}\dot{H}\right)\mu_I(t,x) - \frac{1}{a^2(t)}\partial^2\mu_I(t,x) = 0$$

and quantizing μ_I where

$$H = \frac{\dot{a}}{a}\,, \quad \mu = a^{3/2}\chi\,. \tag{15}$$

With "appropriate" initial conditions and using slow-roll approximation one can show that solution to μ_I in the Fourier space is

$$\mu_I(k,t) = \sqrt{\frac{\pi}{4H}} H_\nu^{(1)} \left(\frac{k}{aH} \right) \tag{16}$$

where $H_\nu^{(1)}$ is the Hankel function of the first type and

$$\nu = \left(\frac{9}{4} - \frac{m_\chi^2 + \kappa^2 \bar{\phi}^2}{H^2} \right)^{1/2} . \tag{17}$$

Then the tree level two point function is

$$\langle \chi_{I,1} \chi_{I,2} \rangle = \int \frac{d^3k}{(2\pi)^3} \frac{e^{i\vec{k}\cdot(\vec{x_1} - \vec{x_2})}}{a^3} \left| \mu_k(t) \right|^2 . \tag{18}$$

In the UV limit by using the asymptotic behaviour of the Hankel function one can see $\mu_k \to \frac{1}{\sqrt{k}}$ and hence the integral diverges as might one expect.

Let us now demonstrate the adiabatic regularization by regularizing this integral. Main idea is to make an expansion in derivatives of $a(t)$, i.e.

$$\frac{\dot{a}}{a} \sim \mathcal{O}(1); \quad \left(\frac{\dot{a}}{a}\right)^2, \frac{\ddot{a}}{a} \sim \mathcal{O}(2)$$

so on. The physical motivation is that, there should be no creation of particles whose energies are much larger then the space-time curvature i.e. when

$$\frac{k^2}{a^2(t)} + m^2 \gg \left(\frac{\dot{a}}{a}\right)^2, \frac{\ddot{a}}{a}\,.$$

It will prove to easier to do this if we define

$$\mu_k = \frac{1}{\sqrt{2W_k}} e^{-i\int^t W_k(t')dt'} \tag{19}$$

Equation becomes:

$$-\frac{1}{2}\frac{\ddot{W}_k}{W_k} + \frac{3}{4}\frac{\dot{W}_k^2}{W_k^2} - W_k^2 + \left(\frac{k^2}{a^2} + m_\chi^2 + \kappa^2\bar{\phi}^2 - \frac{9}{4}H^2\right) = 0$$

Now solve this in adiabatic orders:

$$W_k^{(0)} = \sqrt{\frac{k^2}{a^2} + m_\chi^2 + \kappa^2\bar{\phi}^2} \equiv \omega_k$$
$$W_k^{(1)} = 0$$
$$W_k^{(2)} = \frac{H^2}{2\omega_k}\left(-\frac{9}{4} - \left(\frac{k}{a\omega_k}\right)^2 + \frac{5}{4}\left(\frac{k}{a\omega_k}\right)^4\right)$$

Let us now expand our expectation value in the adiabatic orders:

$$\langle \chi_{I,1}\chi_{I,2}\rangle = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}}{a^3}\frac{1}{2W_k(t)} = \int \Box dkk^2\frac{1}{\omega_k}\left(1-\frac{W_k^{(2)}}{\omega_k}+\cdots\right) \tag{20}$$

where we have only typed the UV diverging parts. Subtle part here is that, if an adiabatic term $W_k^{(n)}$ gives a diverging contribution, we should subtract it as a whole, not only its diverging parts. So as a result we write down the regularized two point function as:

$$\langle \chi_{I,1}\chi_{I,2}\rangle_{\text{reg}} = \int \frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot(\vec{x_1}-\vec{x_2})}}{a^3}\left(\frac{\pi}{4H}\left|H_\nu^{(1)}\left(\frac{k}{aH}\right)\right|^2-\frac{1}{2\omega_k}+\frac{W_k^{(2)}}{\omega_k^2}\right) \tag{21}$$

As another application of adiabatic regularization, in the paper (A.Kaya, ESK; arXiv:1509.00489 [gr-qc]) we calculate one-loop effective potential coming from the field χ , by integrating out the modes χ in the path integral to get

$$V_{eff}(\phi) = \frac{1}{2a(t)^3}\, g^2\, \phi^2 \int \frac{d^3k}{(2\pi)^3} \int_0^1 ds\ |\mu_k[t,s,\phi]|^2 \,.$$

where s is a parameter we have introduced by replacing $\kappa^2 \to s\kappa^2$. This effective potential goes indeed to Coleman-Weinberg potential for the flat case. The procedure of adiabatic regularization can be used here to calculate

$$V_{eff}(\phi) = \frac{1}{2a(t)^3}\, g^2\, \phi^2 \int \frac{d^3k}{(2\pi)^3} \int_0^1 ds\ \left[|\mu_k[t,s,\phi]|^2 - \left|\mu_k^{ad(2)}\right|^2 \right],$$

where

$$\left|\mu_k^{ad(2)}\right|^2 = \frac{1}{2\omega_k}\left[1 + \frac{9}{8}\frac{H^2}{\omega_k^2} + \frac{3}{4}\frac{\dot{H}}{\omega_k^2} - \frac{5}{8}\frac{H^2k^4}{a^4\omega_k^6} - \frac{1}{4}\frac{\dot{H}k^2}{a^2\omega_k^4} + \frac{1}{2}\frac{H^2k^2}{a^2\omega_k^4}\right].$$

For this case the subtracted terms in the effective potential can be thought to arise from a counterterm potential $\delta V(\phi)$ appearing in the bare action. It can be found as

$$\delta V(\phi) = -a_1\kappa^4\phi^4 - 2a_1\kappa^2 m_\chi^2\phi^2 - \kappa^2\phi^2\left[a_2H^2 + a_3\dot{H}\right]$$

where

$$a_1 = \frac{1}{16\pi^2}\int_0^\infty \frac{k^2dk}{(k^2+1)^{1/2}}, \quad a_2 = \frac{9}{64\pi^2}\int_0^\infty \frac{k^2\,dk}{(k^2+1)^{3/2}} + \cdots$$

This interpretation justifies the adiabatic subtraction terms since the regularization procedure can be recast as a standard renormalization method.

Thank You!