Generalized tracker Quintessence models for dark energy

Nandan Roy
The Institute for Fundamental Study
Naresuan University, Thailand

Cosmology from Home 2020
24 August 2020 – 04 September 2020

Urena-Lopez, L. Arturo, and Nandan Roy. "Generalized tracker Quintessence models for dark energy." **arXiv:2007.08873 (2020)**.

Cosmic Coincidence Problem

The standard model of cosmology suffers from a key problem explaining why the dark energy density is comparable to the matter density today. Both in the remote past and in the far future the ratio of these two densities may be of the order of several hundred order of magnitude but why the ratio is of the order of one today. Are we living in some special time? This is called cosmic coincidence problem.

Another problem is why the energy density of the dark energy is tiny compared to the particle physics scale? To achieve a present universe the value of the cosmological parameters has to be extremely fine tuned. The fine tuning problem.

Tracking Solutions

Introduction of tracker solution can help us to alleviate the cosmic coincidence problem and the fine tuning problem.

The "tracker fields," [1] are a form of scalar field in which the field roll down a potential according to an attractor like solution to the equations of motion. The tracker solution is an attractor solution in that a very wide range of initial conditions approach a common evolutionary track, so that the cosmology is insensitive to the initial conditions. The advantage of tracking behaviour is that it can converge a large number of solutions with different initial condition and hence wiping out memory of the initial conditions from the solutions.

1. Steinhardt, Paul J., Limin Wang, and Ivaylo Zlatev. "Cosmological tracking solutions." *Physical Review D* 59.12 (1999): 123504.

Tracking Solutions

Introduction of tracker solution can help us to alleviate the cosmic coincidence problem and the fine tuning problem.

The "tracker fields," [1] are a form of scalar field in which the field roll down a potential according to an attractor like solution to the equations of motion. The tracker solution is an attractor solution in that a very wide range of initial conditions approach a common evolutionary track, so that the cosmology is insensitive to the initial conditions. The advantage of tracking behaviour is that it can converge a large number of solutions with different initial condition and hence wiping out memory of the initial conditions from the solutions.

For the potentials explored in the literature, it was concluded that tracker potentials were unable to provide the accelerated expansion of the Universe that is required by cosmological observations

In this work we have formulated a new generalized way to identify the tracking solutions for a large class of quintessence potentials by considering a general parametrization of potentials which can have late time behaviour consistent with the observations. We have also consider linear perturbations of the quintessence field with the same general parametrization and its influence on the cosmological parameters.

1. Steinhardt, Paul J., Limin Wang, and Ivaylo Zlatev. "Cosmological tracking solutions." *Physical Review D* 59.12 (1999): 123504.

Theoretical Background

A homogeneous and isotropic universe is mathematically represented by FRW metric,

$$ds^2 = cdt^2 - a^2(t)\left[\frac{dr^2}{1-kr^2} + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right)\right]$$

For a spatially flat universe $k = 0$.

$$S = \int \mathrm{d}^4x\sqrt{-g}\left[\frac{1}{2\kappa^2}R + \mathcal{L}_\phi\right] + S_M, \quad \mathcal{L}_\phi = -\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)$$

Theoretical Background

$$H^2 = \frac{\kappa^2}{3}\left(\sum_j \rho_j + \rho_\phi\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\sum_j (\rho_j + p_j) + (\rho_\phi + p_\phi)\right]$$

$$\dot{\rho}_j = -3H\,(\rho_j + p_j)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi}$$

- H → Hubble Parameter
- ρ_j → Density of the components
- p_j → Pressure of the components
- ϕ → Scalar field
- $V(\phi)$ → Potential

Density of the scalar field

$$\rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi)$$

Pressure of the scalar field

$$p_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

EOS of the scalar field

$$w_\phi = \frac{P_\phi}{\rho_\phi}$$

$$w_\phi = \gamma_\phi - 1$$

Arbitrariness of the Quintessence Scalar Field Potentials

$$H^2 = \frac{\kappa^2}{3}\left(\sum_j \rho_j + \rho_\phi\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\sum_j (\rho_j + p_j) + (\rho_\phi + p_\phi)\right]$$

$$\dot{\rho}_j = -3H\left(\rho_j + p_j\right)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi}$$

- H → Hubble Parameter
- ρ_j → Density of the components
- p_j → Pressure of the components
- ϕ → Scalar field
- $V(\phi)$ → Potential

$V(\phi)$ → *The Potential is Arbitrary*

The Dynamical System

$$H^2 = \frac{\kappa^2}{3}\left(\sum_j \rho_j + \rho_\phi\right)$$

$$\dot{H} = -\frac{\kappa^2}{2}\left[\sum_j (\rho_j + p_j) + (\rho_\phi + p_\phi)\right]$$

$$\dot{\rho}_j = -3H\,(\rho_j + p_j)$$

$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi}$$

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H} = \Omega_\phi^{1/2}\sin(\theta_\phi/2), \quad y = \frac{\kappa V_\phi^{1/2}}{\sqrt{3}H} = \Omega_\phi^{1/2}\cos(\theta_\phi/2)$$

$$y_1 \equiv -2\sqrt{2}\frac{\partial_\phi V_\phi^{1/2}}{H}, \quad y_2 \equiv -4\sqrt{3}\frac{\partial_\phi^2 V_\phi^{1/2}}{\kappa H}$$

To write the EOM as a set of autonomous dynamical systems we have used the above polar transformations.

The Dynamical System

$$H^2 = \frac{\kappa^2}{3}\left(\sum_j \rho_j + \rho_\phi\right)$$
$$\dot{H} = -\frac{\kappa^2}{2}\left[\sum_j (\rho_j + p_j) + (\rho_\phi + p_\phi)\right]$$
$$\dot{\rho}_j = -3H(\rho_j + p_j)$$
$$\ddot{\phi} = -3H\dot{\phi} - \frac{dV(\phi)}{d\phi}$$

Dimensionless Transformations

$$x = \frac{\kappa\dot{\phi}}{\sqrt{6}H} = \Omega_\phi^{1/2}\sin(\theta_\phi/2), \quad y = \frac{\kappa V_\phi^{1/2}}{\sqrt{3}H} = \Omega_\phi^{1/2}\cos(\theta_\phi/2)$$
$$y_1 \equiv -2\sqrt{2}\frac{\partial_\phi V_\phi^{1/2}}{H}, \quad y_2 \equiv -4\sqrt{3}\frac{\partial_\phi^2 V_\phi^{1/2}}{\kappa H}$$

$$\theta' = -3\sin\theta + y_1$$
$$y_1' = \frac{3}{2}\gamma_{tot}\, y_1 + \Omega_\phi^{1/2}\sin(\theta/2) y_2$$
$$\Omega_\phi' = 3(\gamma_{tot} - \gamma_\phi)\Omega_\phi$$

A prime is derivative w.r.t the *e*-folding $N = \ln a$

The energy density of the scalar field: $\Omega_\phi = \kappa^2\rho_\phi/(3H^2)$

EOS of scalar field: $\gamma_\phi = (p_\phi + \rho_\phi)/\rho_\phi = 1 - \cos\theta$

General Parameterization of Potentials

There are two different ways to specify y_2

- Find out y_2 for a given potential.
- Considering a particular form of y_2 and find out the potential.

$V(\phi) \rightarrow$ [equations] $\rightarrow y_2$

$y_2 \rightarrow$ [equations] $\rightarrow V(\phi)$

$$y = \frac{\kappa V_\phi^{1/2}}{\sqrt{3} H}$$

$$y_1 \equiv -2\sqrt{2}\frac{\partial_\phi V_\phi^{1/2}}{H}$$

$$y_2 \equiv -4\sqrt{3}\frac{\partial_\phi^2 V_\phi^{1/2}}{\kappa H}$$

General Parameterization of Potentials

$$V(\phi) \longrightarrow y_2$$

Roy, Nandan, Alma X. Gonzalez-Morales, and L. Arturo Ureña-López. Physical Review D 98.6 (2018): 063530.

Potential $V(\phi)$	Closed form of y_2
$A^4(1+B\phi)^{2\lambda}$	$\frac{1-\lambda}{2\lambda}\, y_1^2/y$
$A^4 \exp\left(-\phi^2/\lambda^2\right)$	$\frac{12}{\kappa^2\lambda^2} y - \frac{1}{2} y_1^2/y$
$A^4[1+\cos(\phi/\lambda)]$	$\frac{3}{\kappa^2\lambda^2} y$
$A^{4+\lambda}\phi^{-\lambda}$	$-\frac{1}{\lambda}(\frac{\lambda}{2}+1) y_1^2/y$
$A^4 e^{2\lambda\kappa^2\phi^2}$	$-24\lambda y - \frac{1}{2} y_1^2/y$
$A^4(1-e^{-\lambda\kappa\phi})^2$	$-\sqrt{6}\lambda y_1$
$A^4 \cosh(\lambda\kappa\phi)$	$-6\lambda^2 y + \frac{1}{2} y_1^2/y$
$A^4[\cosh(\lambda\kappa\phi)]^{-1}$	$6\lambda^2 y - \frac{3}{2} y_1^2/y$
$A^4[\sinh(\lambda_1\kappa\phi)]^{-\lambda_2}$	$6\lambda_1^2\lambda_2 y - (1/\lambda_2)(1+\lambda_2/2) y_1^2/y$
$A^4\left[e^{\lambda_1\kappa\phi} + e^{\lambda_2\kappa\phi}\right]$	$6\lambda_1\lambda_2 y + \sqrt{6}(\lambda_1+\lambda_2) y_1 + \frac{1}{2} y_1^2/y$

All y_2 follow a general mathematical form.

$$y_2 = y \sum_{i=0}^{n} \alpha_i \left(\frac{y_1}{y}\right)^i$$

n=2 is enough to get back all the potentials

General Parameterization of Potentials

$y_2 \longrightarrow V(\phi)$

$$y_2 = y \sum_{i=0}^{n} \alpha_i \left(\frac{y_1}{y}\right)^i$$

$$y = \frac{\kappa V_\phi^{1/2}}{\sqrt{3}H}, \; y_1 \equiv -2\sqrt{2}\frac{\partial_\phi V_\phi^{1/2}}{H},$$

$$y_2 \equiv -4\sqrt{3}\frac{\partial_\phi^2 V_\phi^{1/2}}{\kappa H}$$

$$y_2 = y\left(\alpha_0 + \alpha_1 y_1/y + \alpha_2 y_1^2/y^2\right)$$

No	Structure of y_2/y	Form of the potentials $V(\phi)$
Ia	$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 \neq -\frac{1}{2}$	$(A + B\phi)^{\frac{2}{(2\alpha_2+1)}}$
Ib	$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -\frac{1}{2}$	$A^2 e^{2B\phi}$
IIa	$\alpha_0 \neq 0, \alpha_1 = 0, \alpha_2 \neq -\frac{1}{2}$	$A^2 \cos\left[\sqrt{\alpha_0\kappa^2(1+2\alpha_2)}(\phi - B)/2\sqrt{3}\right]^{\frac{2}{1+2\alpha_2}}$
IIb	$\alpha_0 \neq 0, \alpha_1 = 0, \alpha_2 = -\frac{1}{2}$	$A^2 \exp\left(-\kappa^2\alpha_0\phi^2/12\right)\exp(2B\phi)$
IIIa	$\alpha_0 = 0, \alpha_1 \neq 0, \alpha_2 \neq -\frac{1}{2}$	$\left[A\exp\left(\alpha_1\kappa\phi/\sqrt{6}\right) + B\right]^{\frac{2}{1+2\alpha_2}}$
IIIb	$\alpha_0 = 0, \alpha_1 \neq 0, \alpha_2 = -\frac{1}{2}$	$A^2 \exp\left[2B\ \exp\left(\kappa\alpha_1\phi/\sqrt{6}\right)\right]$
IVa	$\alpha_0 \neq 0, \alpha_1 \neq 0, \alpha_2 \neq -\frac{1}{2}$	$A^2\exp\left(\frac{\kappa\alpha_1\phi}{\sqrt{6}(1+2\alpha_2)}\right)\left\{\cos\left[\left(-\frac{\kappa^2\alpha_1^2}{24} + \frac{\kappa^2\alpha_0}{12}(1+2\alpha_2)\right)^{\frac{1}{2}}(\phi - B)\right]\right\}^{\frac{2}{1+2\alpha_2}}$
IVb	$\alpha_0 \neq 0, \alpha_1 \neq 0, \alpha_2 = -\frac{1}{2}$	$A^2\exp\left[\frac{\kappa\alpha_0\phi}{\sqrt{6}\alpha_1} + 2B\exp\left(\frac{\kappa\alpha_1\phi}{\sqrt{6}}\right)\right]$

General Parameterization of Potentials

No	Structure of y_2/y	Form of the potentials $V(\phi)$
Ia	$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 \neq -\frac{1}{2}$	$(A + B\phi)^{\frac{2}{(2\alpha_2+1)}}$
Ib	$\alpha_0 = 0, \alpha_1 = 0, \alpha_2 = -\frac{1}{2}$	$A^2 e^{2B\phi}$
IIa	$\alpha_0 \neq 0, \alpha_1 = 0, \alpha_2 \neq -\frac{1}{2}$	$A^2 \cos\left[\sqrt{\alpha_0 \kappa^2 (1+2\alpha_2)}(\phi - B)/2\sqrt{3}\right]^{\frac{2}{1+2\alpha_2}}$
IIb	$\alpha_0 \neq 0, \alpha_1 = 0, \alpha_2 = -\frac{1}{2}$	$A^2 \exp\left(-\kappa^2 \alpha_0 \phi^2/12\right) \exp(2B\phi)$
IIIa	$\alpha_0 = 0, \alpha_1 \neq 0, \alpha_2 \neq -\frac{1}{2}$	$\left[A \exp\left(\alpha_1 \kappa\phi/\sqrt{6}\right) + B\right]^{\frac{2}{1+2\alpha_2}}$
IIIb	$\alpha_0 = 0, \alpha_1 \neq 0, \alpha_2 = -\frac{1}{2}$	$A^2 \exp\left[2B \ \exp\left(\kappa\alpha_1\phi/\sqrt{6}\right)\right]$
IVa	$\alpha_0 \neq 0, \alpha_1 \neq 0, \alpha_2 \neq -\frac{1}{2}$	$A^2 \exp\left(\frac{\kappa\alpha_1\phi}{\sqrt{6}(1+2\alpha_2)}\right)\left\{\cos\left[\left(-\frac{\kappa^2\alpha_1^2}{24} + \frac{\kappa^2\alpha_0}{12}(1+2\alpha_2)\right)^{\frac{1}{2}}(\phi - B)\right]\right\}^{\frac{2}{1+2\alpha_2}}$
IVb	$\alpha_0 \neq 0, \alpha_1 \neq 0, \alpha_2 = -\frac{1}{2}$	$A^2 \exp\left[\frac{\kappa\alpha_0\phi}{\sqrt{6}\alpha_1} + 2B \exp\left(\frac{\kappa\alpha_1\phi}{\sqrt{6}}\right)\right]$

Potential $V(\phi)$	Closed form of y_2
$A^4(1+B\phi)^{2\lambda}$	$\frac{1-\lambda}{2\lambda}\, y_1^2/y$
$A^4 \exp\left(-\phi^2/\lambda^2\right)$	$\frac{12}{\kappa^2\lambda^2} y - \frac{1}{2} y_1^2/y$
$A^4[1+\cos(\phi/\lambda)]$	$\frac{3}{\kappa^2\lambda^2} y$
$A^{4+\lambda}\phi^{-\lambda}$	$-\frac{1}{\lambda}(\frac{\lambda}{2}+1) y_1^2/y$
$A^4 e^{2\lambda\kappa^2\phi^2}$	$-24\lambda y - \frac{1}{2} y_1^2/y$
$A^4(1-e^{-\lambda\kappa\phi})^2$	$-\sqrt{6}\lambda y_1$
$A^4 \cosh(\lambda\kappa\phi)$	$-6\lambda^2 y + \frac{1}{2} y_1^2/y$
$A^4[\cosh(\lambda\kappa\phi)]^{-1}$	$6\lambda^2 y - \frac{3}{2} y_1^2/y$
$A^4[\sinh(\lambda_1\kappa\phi)]^{-\lambda_2}$	$6\lambda_1^2\lambda_2 y - (1/\lambda_2)(1+\lambda_2/2) y_1^2/y$
$A^4\left[e^{\lambda_1\kappa\phi} + e^{\lambda_2\kappa\phi}\right]$	$6\lambda_1\lambda_2 y + \sqrt{6}(\lambda_1+\lambda_2) y_1 + \frac{1}{2} y_1^2/y$

The general Form

$$y_2 = y\left(\alpha_0 + \alpha_1 y_1/y + \alpha_2 y_1^2/y^2\right)$$

General Tracker Solution

With the parametrization of $y_2 = y\left(\alpha_0 + \alpha_1 y_1/y + \alpha_2 y_1^2/y^2\right)$ it is possible to easily write down the physical solutions.

$$\begin{aligned} \theta' &= -3\sin\theta + y_1 \\ y_1' &= \frac{3}{2}\gamma_{tot} y_1 + \Omega_\phi^{1/2}\sin(\theta/2) y_2 \\ \Omega_\phi' &= 3\left(\gamma_{tot} - \gamma_\phi\right)\Omega_\phi \end{aligned}$$

The critical condition of the system can be easily found from the equations:

$$y_{1c} = 3\sin\theta_c$$

$$\left(\gamma_{tot} + \frac{\alpha_0}{9}\Omega_{\phi c} + \frac{\sqrt{2}}{3}\alpha_1\Omega_{\phi c}^{1/2}\gamma_{\phi c}^{1/2} + 2\alpha_2\gamma_{\phi c}\right)\sin\theta_c = 0$$

$$(\gamma_{tot} - \gamma_{\phi c})\Omega_{\phi c} = 0$$

There can be four classes of critical points:

A. Fluid dominated solution $\Omega_{\phi_c} = 0,\ \gamma_c = 0, 2$

B. Scaling solution $\gamma_{\phi c} = \gamma_{tot}, \Omega_\phi \propto \gamma_{tot}$

C. Tracking solution $\gamma_{\phi c} \propto \gamma_{tot}$

D. Quintessence dominated solution $\gamma_{tot} = \gamma_{\phi c}, \Omega_\phi = 1$

General Tracker Solution

$$y_{1c} = 3\sin\theta_c$$

$$\left(\gamma_{tot} + \frac{\alpha_0}{9}\Omega_{\phi c} + \frac{\sqrt{2}}{3}\alpha_1\Omega_{\phi c}^{1/2}\gamma_{\phi c}^{1/2} + 2\alpha_2\gamma_{\phi c}\right)\sin\theta_c = 0$$

$$(\gamma_{tot} - \gamma_{\phi c})\Omega_{\phi c} = 0$$

Our main interest to study the case when $\alpha_0 = 0 = \alpha_1$, this corresponds to a critical condition:

$$\boxed{\gamma_{\phi c} = -\gamma_{tot}/(2\alpha_2)}$$

In the literature this is known as the tracking solution in which the EOS of the scalar field track the background EOS.

Given that we expect $0 < \gamma_{\phi c} < 2$, we find that the critical condition exists for $0 < -\gamma_{tot}/\alpha_2 < 4$, and then $\alpha_2 < -\gamma_{tot}/4$

If we want the tracking to start at times during radiation domination then $\gamma_{tot} = 4/3$, then the upper bound in the active parameter is $\alpha_2 < -1/3$

General Tracker Solution

For example

- The potential is power-law quintessence $V(\phi) = M^{4-p}\phi^p$ $\alpha_0 = \alpha_1 = 0$
- Which corresponds to $p = 2/(1+2\alpha_2)$
- The tracker condition translates into
  - $\gamma_{\phi c} = p\,\gamma_{tot}/(p-2)$; $\alpha_2 < -1/3$
  - The existence condition of the tracker solution for power-law potentials is $p > 6$ (for $-1/2 < \alpha_2 < -1/3$) and $p < 0$ (for $\alpha_2 < -1/2$), whereas $\alpha_2 = -1/2$ corresponds to an exponential potential

General Tracker Solution

For example

- The potential is power-law quintessence $V(\phi) = M^{4-p}\phi^p$ $\alpha_0 = \alpha_1 = 0$
- Which corresponds to $p = 2/(1 + 2\alpha_2)$
- The tracker condition translates into
  - $\gamma_{\phi c} = p\,\gamma_{tot}/(p-2)$; $\alpha_2 < -1/3$
  - The existence condition of the tracker solution for power-law potentials is $p > 6$ (for -½ < α_2 < -⅓) and p < 0 (for α_2 < -½), whereas α_2 = -½ corresponds to an exponential potential

- ❑ Tracker condition can be satisfied approximately by a more general class of potentials, with $\alpha_0 \neq 0$ $\alpha_1 \neq 0$
- ❑ As long as the quintessence density parameter Ω_ϕ is negligible with respect to the dominant one, which is regularly the case at early times in the evolution of the Universe.

Linear Perturbation

Let us now consider the case of linear perturbations φ of the quintessence field in the form $\phi(x,t) = \phi(t) + \varphi(x,t)$

We chose the synchronous gauge, $ds^2 = -dt^2 + a^2(t)(\delta_{ij} + h_{ij})dx^i dx^j$

The linearized Klein-Gordon equation for a given Fourier mode $\varphi(k,t)$

$$\ddot{\varphi} = -3H\dot{\varphi} - \left(\frac{k^2}{a^2} + \partial_\phi^2 V\right)\varphi - \frac{1}{2}\dot{\phi}\dot{\bar{h}}$$

where a dot means derivative with respect the cosmic time, $\bar{h} = h^j{}_j$ of the tensor metric perturbations and k is a comoving wavenumber.

Linear Perturbation

We consider the following transformations to write the linearized KG equation to make a dynamical system

$$\sqrt{\frac{2}{3}}\frac{\kappa\dot{\varphi}}{H} \equiv -\Omega_\phi^{1/2} e^\beta \cos(\vartheta/2)\,, \quad \frac{\kappa y_1 \varphi}{\sqrt{6}} \equiv -\Omega_\phi^{1/2} e^\beta \sin(\vartheta/2)$$

And also with another set of transformations

$$\delta_0 = -e^\beta \sin(\theta/2 - \vartheta/2)$$

$$\delta_1 = -e^\beta \cos(\theta/2 - \vartheta/2)$$

The linearized KG equation is transformed to the following dynamical system

$$\delta_0' = -\left[3\sin\theta + \frac{k^2}{k_J^2}(1-\cos\theta)\right]\delta_1 + \frac{k^2}{k_J^2}\sin\theta\delta_0 - \frac{\bar{h}'}{2}(1-\cos\theta)$$

$$\delta_1' = -\left(3\cos\theta + \frac{k_{eff}^2}{k_J^2}\sin\theta\right)\delta_1 + \frac{k_{eff}^2}{k_J^2}(1+\cos\theta)\,\delta_0 - \frac{\bar{h}'}{2}\sin\theta$$

Jeans wavenumber $k_J^2 \equiv a^2 H^2 y_1$; $\quad k_{eff}^2 \equiv k^2 - \frac{y_2}{2y} a^2 H^2 \Omega_\phi$

$$\partial_\phi^2 V = H^2 (y_1^2/4 - y y_2/2)$$

$$\delta\rho_\phi/\rho_\phi = (\dot{\phi}\dot{\varphi} + \partial_\phi V \varphi)/\rho_\phi = \delta_0$$

Numerical Solution

Using the tracker condition $\gamma_{\phi c} = -\gamma_{tot}/(2\alpha_2)$ and the equating the solution of the background EOM at the radiation matter equality a set of initial condition can be approximate

Initial Condition for background

$$\cos\theta_i = 1 + \frac{2}{3\alpha_2}$$

$$y_{1i} = 3\sin\theta_i$$

$$\Omega_{\phi i} = A \times a_i^{4(1+1/2\alpha_2)} \left(\frac{\Omega_{m0}}{\Omega_{r0}}\right)^{1+1/2\alpha_2} \Omega_{\phi 0}$$

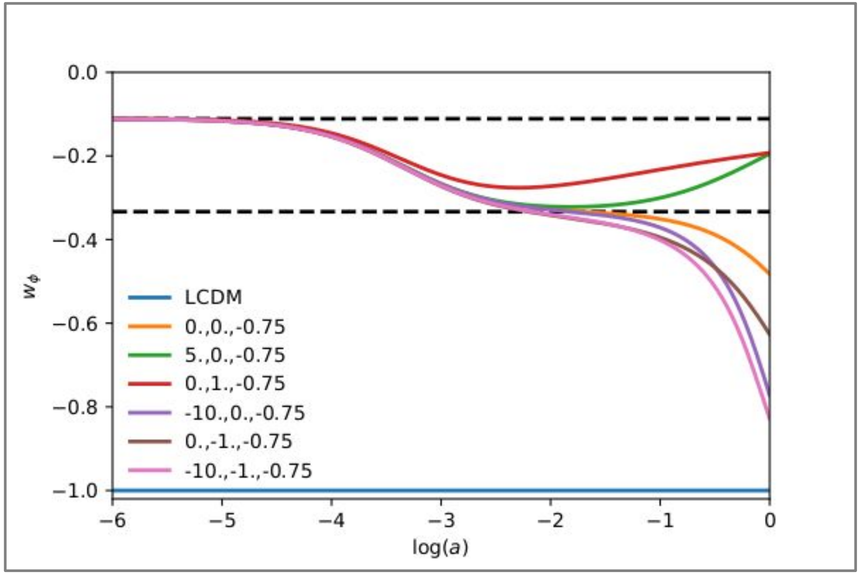
Initial condition for perturbation

$$\delta_{0i} = 0$$

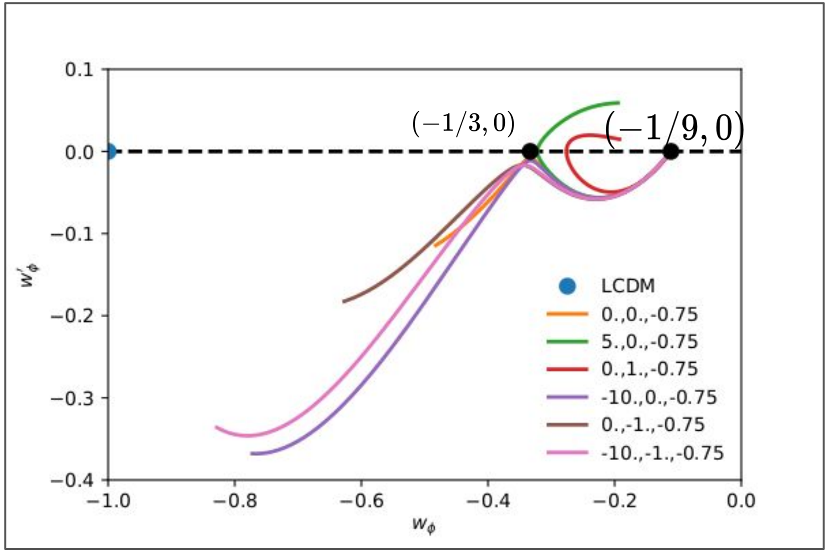
$$\delta_{1i} = 0$$

These initial condition together with the background and perturbation equations has been implemented in a amended version of the Boltzman code CLASS.

Numerical Solution (with the linear density perturbation)

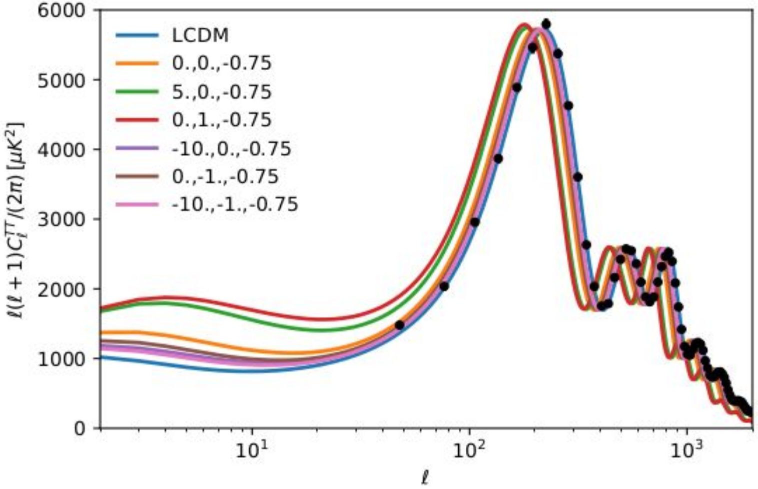


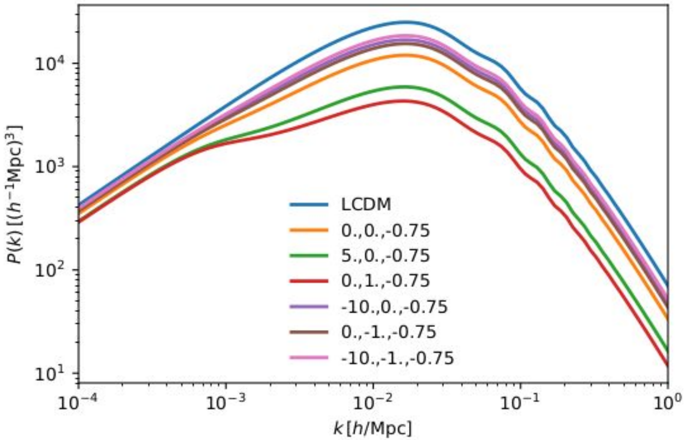
The evolution of the EoS for tracker quintessence models. Each curve represents a model with the indicated values of the triplet alpha parameters . The first example in all figures corresponds to the inverse power-law potential.



Phase space of the EoS , for the same cases (with the same colors) as in the other plot. The blue dot represents the cosmological constant case, whereas the black dots represent the aforementioned tracker values.

Numerical Solution (with the linear density perturbation)



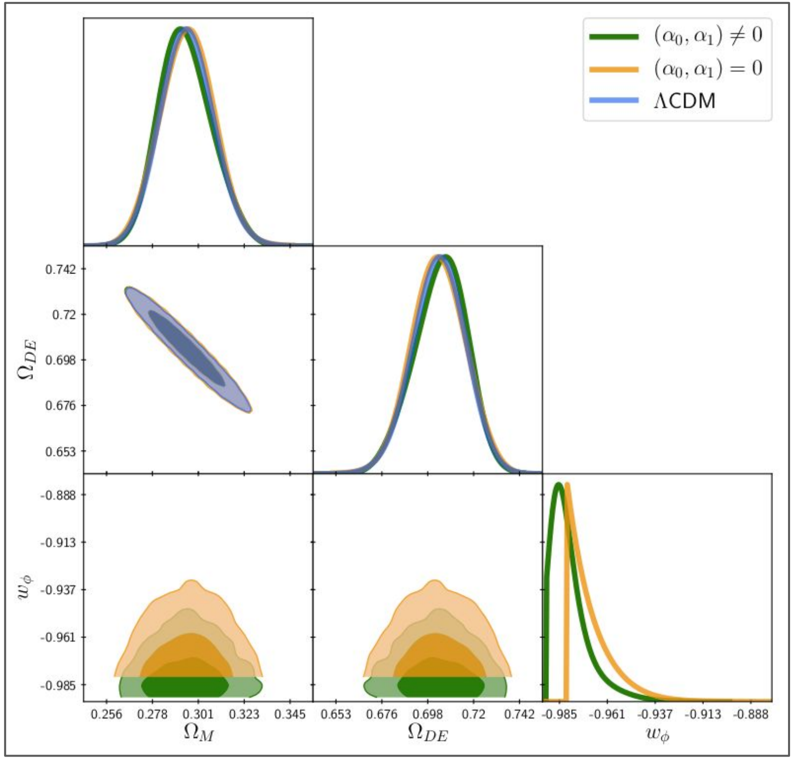


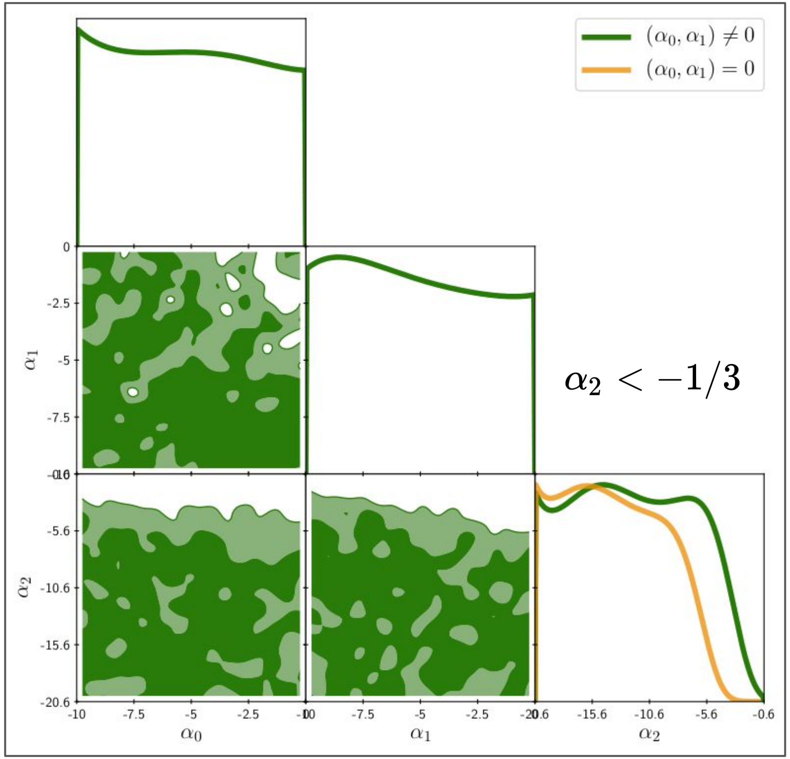
Plots of the two-point temperature autocorrelation power spectrum of the Cosmic Microwave Background (CMB) and the mass power spectrum (MPS) of linear density perturbations P(k), for the same numerical examples. The cases in which the EoS deviates the most from the accelerating regime also show the major differences in the CMB anisotropies and the MPS with respect to the ΛCDM results.

The active parameters α_0 and α_1 help to close the gaps with respect to ΛCDM if they take on negative values.

Comparison with Observations

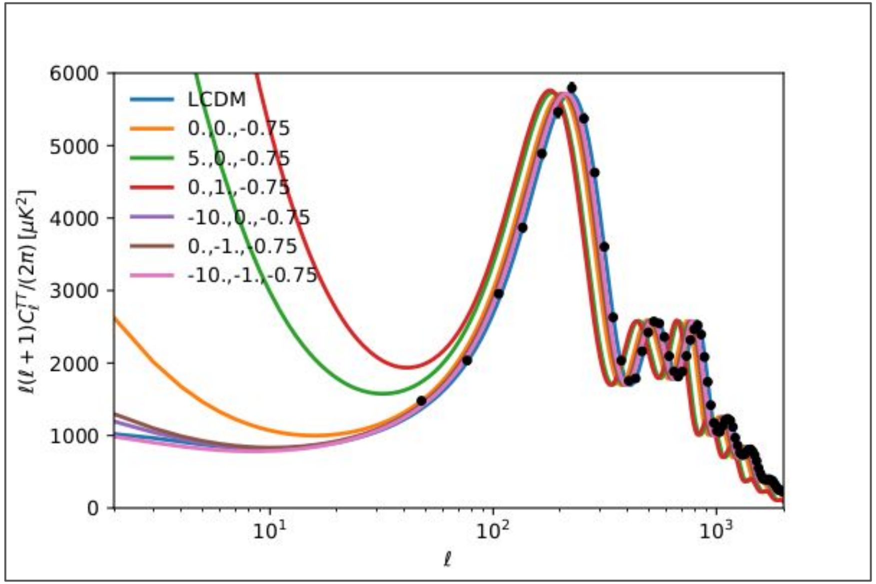
We use the aforementioned amended version of the Boltzmann CLASS and the Monte Carlo code MONTE PYTHON (v3.2). We consider two data sets that are sensitive to the background dynamics: (i) the Pantheon supernova data, (ii) BAO (baryonic acoustic oscillations) measurements, together with a Planck2018 prior on the baryonic matter component: $\omega_b = 0.02230 \pm 0.00014$

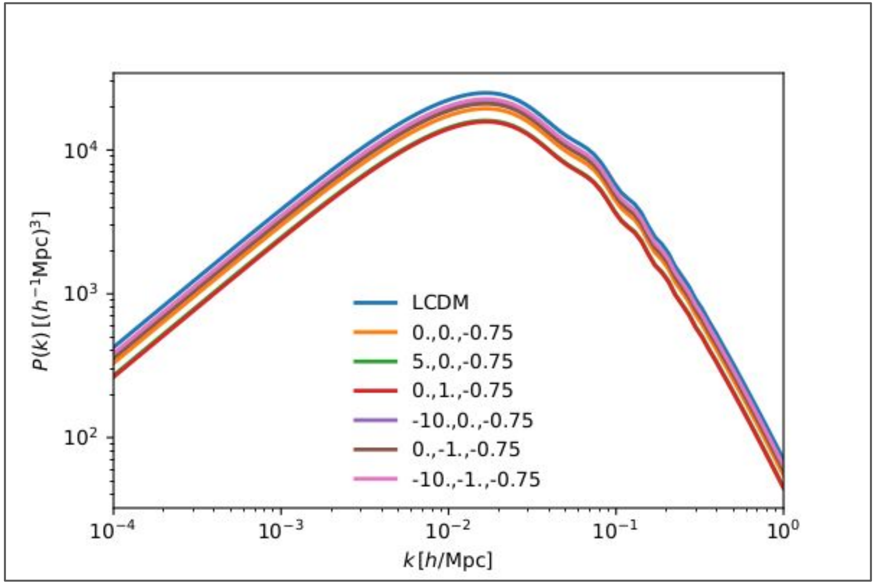




$\alpha_2 < -1/3$

Numerical Solution (without the linear density perturbation)





To highlight the importance of linear density perturbations in DE studies, we show the same cosmological quantities as previous figure, for the same tracker quintessence models, but without the linear density perturbations from the quintessence field.

A quick comparison between Figs with and without quintessence perturbations show that one can obtain misleading constraints on the quintessence models if density perturbations are neglected. In particular at low multipoles.

Conclusion

- We have revised the case of tracker quintessence models of DE using a parameterization including the influence of linear perturbations of the quintessence field.
- The new formalism allowed the identification of new tracker models which are able to have a late time behavior more similar to that of the cosmological constant of ΛCDM.
- We identified the necessary condition to be satisfied if a quintessence potential is to have a tracker behavior; the condition involves just one of the so-called active parameters in the parameterization of the potential.
- The other active parameters play a role in the late time dynamics of the quintessence field, and we showed that the latter behaves more similar to the cosmological constant if they take on negative values.
- Even though the common wisdom is to neglect DE density perturbations, we showed that they should be included for full consistency in the calculation of observables. In the particular case of tracker potentials, it was clear that density perturbations are important also to have more stringent constraints on the parameters of the models.